M-Polynomial of Some Operations of Path and K-Banhatti Indices

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Abstract
Among the introduced graph algebraic polynomials, one of the most intriguing polynomials is M-Polynomial, which is a unified way tool to compute degree-based topological indices. Graph operations are important in many applications of graph theory, because we can generate huge graphs from small graphs by using graph operations. Till now, many researchers compute degree-based topological indices of various simple and connected graphs via M-Polynomial approach. However, no one has paid attention to the M-Polynomial of numerous graph operations. In this article, we attempt to compute the M-Polynomial of different graph operations on some paths of different orders. Further, we evaluate the K-Banhatti group of indices for the considered graph operations using M-Polynomial. This article also reports some graphical comparison among the computed indices for the graph operations. The findings of our computations may be useful in locating some buried information in a variety of large graphs.

Keywords: - Some Graph Operations, M-Polynomial, K-Banhatti group of indices.

Subject Classification (MSC 2010): 05C76; 05C31; 05C07; 05C38

1. Introduction:

Let, G be a simple, connected and undirected graph. $V(G)$ and $E(G)$ represents the set of vertices and set of edges of Grespectively, such that $|V(G)| = n$ represents the order of G and $|E(G)| = m$ represents the size of G. The number of edges occurring to a vertex $v \in V(G)$, is its degree and is denoted by d_v. Let, H_1 and H_2 be two disjoint graphs with order n_1, n_2 and size m_1, m_2 respectively. Then, the Join or Complete product of H_1 and H_2 is the graph which has vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2) \cup \{ab | a \in V(H_1), b \in V(H_2)\}$, which is denoted by $H_1 \vee H_2$. The Cartesian product of H_1 and H_2, denoted by $H_1 \boxdot H_2$, is the graph which has vertex set $V(H_1) \times V(H_2)$, where two distinct vertices $(a,\ b)$ and $(a',\ b')$ are adjacent iff $a = a'$ and $(b,\ b') \in E(H_2)$, or $b = b'$ and $(a,\ a') \in E(H_1)$. The Lexicographic product of H_1 and H_2, which is denoted by $H_1 \circ H_2$, which has vertex set $V(H_1) \times V(H_2)$, where two distinct vertices $(a,\ b)$ and $(a',\ b')$ are adjacent iff either $(a,\ a') \in E(H_1)$ or $a = a'$ and $(b,\ b') \in E(H_2)$. The Corona of H_1 and H_2, which is denoted by $H_1 \odot H_2$, is the graph generated by taking one copy of H_1 and n_1 copies of H_2, and then adjusting the ith vertex of H_1 with an edge to each vertex in the ith copy of H_2. More details about these, interested readers can visit [7, 24, 27, 30, 33].

A number of topological indices have been defined in the literature. In 2016 and 2017, Kulli [18, 19] proposed some novel degree-based topological indices namely first K-Banhatti index, second K-Banhatti index, modified first K-Banhatti index, modified second K-Banhatti index, and harmonic K-Banhatti index. Generally, the degree-based topological index (TI) of a graph G is defined as

$$TI(G) = \sum_{ab\in E(G)} f(d_a, d_b).$$

Where, $f(x,\ y)$ is a function that is suitable for the computation such that $x = d_a, y = d_b$ and $ab \in E(G)$.

The first K-Banhatti index of a graph G is defined as

$$B_1(G) = \sum_{ae}(d_a + d_e).$$

The second K-Banhatti index of a graph G is defined as

$$B_2(G) = \sum_{ae}(d_a . d_e).$$

The modified first K-Banhatti index of a graph G is defined as

$$m_{B_1}(G) = \sum_{ae}\left(\frac{1}{d_a+d_e}\right).$$

The modified second K-Banhatti index of a graph G is defined as

$$m_{B_2}(G) = \sum_{ae}\left(\frac{1}{d_a . d_e}\right).$$

The Harmonic K-Banhatti index of a graph G is defined as

$$HB(G) = \sum_{ae}\left(\frac{2}{d_a+d_e}\right).$$

Where, ae means that the vertex a and edge e are incident in the graph G.

For more details about some degree-based topological indices, interested readers can visit [3, 4, 6, 12-15, 17, 23, 25, 26, 29, 31, 32].

If all of these topological indices can be derived from a single equation, that it would be more interesting and, graph algebraic polynomials play such types of roles. PI polynomial [5], Tutte polynomial [10], matching polynomial [11], Schultz polynomial [16], Zang-Zang polynomial [31], and others are examples of graph polynomials. The Hosoya polynomial is the best and most well-known of these polynomials, and it is used to determine distance-based topological indices like the Wiener index [28] and hyper Wiener index [8] of graphs. Inspiring these, Deutsch et al. [9] proposed the M-Polynomial in 2015, which is useful for finding several degree-based topological indices. They defined the M-Polynomial of a graph G as

$$M(G; x, y) = \sum_{\delta\leq a\leq b\leq\Delta} m_{(a,b)}(G)x^a y^b,$$

where $\delta = \min\{d_a | a \in V(G)\}, \Delta = \max\{d_a | a \in V(G)\}$ and $m_{(a,b)}(G)$ is the number of edges $ab \in E(G)$ such that $(\gamma(a), \gamma(b)) = (a, b)$, where $\gamma(a)$ represents the degree of vertex a.

Inspired by this work, several researchers compute M-Polynomial [1, 2, 20-22] for various simple and connected graphs. However, no one has paid attention to the M-Polynomial of numerous graph operations. Therefore, in this article, we attempt to compute the M-Polynomial of several graph operations on paths of different sizes. After computing the M-Polynomials, we recover some important degree-based topological indices known as K-Banhatti group of indices of our considering graphs. We also include some graphical comparison among the computing indices of the considering graphs.

Some important formulae by which we can recover the K-Banhatti group of indices of a graph G using the M-Polynomial of G which is denote as $M(G;x,y)$ are shown in Table-1.

Table 1: Derivation for K-Banhatti group of indices from M-Polynomial

Sl. No.	Topological Index	Derivation from M-Polynomial
1.	First K-Banhatti index, $B_1(G)$	$\left(D_x+D_y+2D_xQ_{-2}J\right)\left(M(G;x,y)\right)_{x=y=1}$
2.	Second K-Banhatti index, $B_2(G)$	$D_xQ_{-2}J\left(D_x+D_y\right)\left(M(G;x,y)\right)_{x=1}$
3.	Modified first K-Banhatti index, $m_{B_1}(G)$	$S_xQ_{-2}J\left(L_x+L_y\right)\left(M(G;x,y)\right)_{x=1}$
4.	Modified second K-Banhatti index, $m_{B_2}(G)$	$S_xQ_{-2}J\left(S_x+S_y\right)\left(M(G;x,y)\right)_{x=1}$
5.	Harmonic K-Banhatti index, $HB(G)$	$2S_xQ_{-2}J\left(L_x+L_y\right)\left(M(G;x,y)\right)_{x=1}$

Where $D_x\left(M(G;x,y)\right)=x\frac{\partial(M(G;x,y))}{\partial x}$, $D_y\left(M(G;x,y)\right)=y\frac{\partial(M(G;x,y))}{\partial y}$, $J\left(M(G;x,y)\right)=M(G;x,y)_{x=y}$,

$S_x\left(M(G;x,y)\right)=\int_0^x\frac{M(G;x,y)|_{x=t}}{t}dt$, $S_y\left(M(G;x,y)\right)=\int_0^y\frac{M(G;x,y)|_{y=t}}{t}dt$, $L_x\left(M(G;x,y)\right)=M(G;x^2,y)$ and

$L_y\left(M(G;x,y)\right)=M(G;x,y^2)$.

2. M-Polynomial and K-Banhatti group of indices of Some Graph Operations

A tree is a connected graph which have n vertices and $n-1$ edges. A cycle C_n is a connected graph which have n vertices, each of whose degree is 2. If we delete one edge from cycle, then we have paths, P_n depending on the order of the path.

2.1. Join or Complete product

Theorem 2.1. For $n>1$, the M-Polynomial of $P_{2n}\vee P_{2n}$ is obtained as

$$M(P_{2n}\vee P_{2n};x,y)=4x^{2n+1}y^{2n+1}+(8n-4)x^{2n+1}y^{2n+2}+\{(2n-2)^2+n\}x^{2n+2}y^{2n+2}.$$

Proof. Based on the degrees, the set of vertices of $P_{2n}\vee P_{2n}$ are partition as:

$$V_{2n+1}=\{v\in V(P_{2n}\vee P_{2n}): d_v=2n+1\},$$

$$V_{2n+2}=\{v\in V(P_{2n}\vee P_{2n}): d_v=2n+2\}.$$

From the graph of $P_{2n}\vee P_{2n}$, we have

$m_{(2n+1,2n+1)}=4, m_{(2n+1,2n+2)}=8n-4$ and $m_{(2n+2,2n+2)}=(2n-2)^2+n$.

Now, by using the definition of M-Polynomial, we have

$M(P_{2n}\vee P_{2n};x,y)=\sum_{\delta\le a\le b\le\Delta}m_{(a,b)}(P_{2n}\vee P_{2n})x^ay^b$,

$= m_{(2n+1,2n+1)}x^{2n+1}y^{2n+1} + m_{(2n+1,2n+2)}x^{2n+1}y^{2n+2} + m_{(2n+2,2n+2)}x^{2n+2}y^{2n+2},$

$= 4x^{2n+1}y^{2n+1} + (8n-4)x^{2n+1}y^{2n+2} + \{(2n-2)^2 + n\}x^{2n+2}y^{2n+2}.$ ■

Theorem 2.2. The K-Banhatti group of indices of $P_{2n} \vee P_{2n}$ are obtained as

1. $B_1(P_{2n} \vee P_{2n}) = 48n^3 + 44n^2 + 52.$
2. $B_2(P_{2n} \vee P_{2n}) = 64n^4 + 112n^3 + 56n^2 + 20.$
3. $m_{B_1}(P_{2n} \vee P_{2n}) = \frac{8}{6n+1} + \frac{4n-2}{3n+1} + \frac{8n-4}{6n+3} + \frac{4n^2-7n+4}{3n+2}.$
4. $m_{B_2}(P_{2n} \vee P_{2n}) = \frac{8}{8n^2+4n} + \frac{1}{4n+1}\left(\frac{8n-4}{2n+1} + \frac{4n-2}{n+1}\right) + \frac{4n^2-7n+4}{(n+1)(4n+2)}.$
5. $HB(P_{2n} \vee P_{2n}) = \frac{16}{6n+1} + \frac{8n-4}{3n+1} + \frac{16n-8}{6n+3} + \frac{8n^2-14n+8}{3n+2}.$

Proof. By using the M-Polynomial of $P_{2n} \vee P_{2n}$ and Table-1, we have

$(D_x + D_y)(M(P_{2n} \vee P_{2n})) = 8(2n+1)x^{2n+1}y^{2n+1} + (8n-4)(4n+3)x^{2n+1}y^{2n+2} + (4n+4)(4n^2 - 7n + 4)x^{2n+2}y^{2n+2},$

$(L_x + L_y)(M(P_{2n} \vee P_{2n})) = 4x^{4n+2}y^{2n+1} + (8n-4)x^{4n+2}y^{2n+2} + (4n^2-7n+4)x^{4n+4}y^{2n+2} + 4x^{2n+1}y^{4n+2} + (8n-4)x^{2n+1}y^{4n+4} + (4n^2-7n+4)x^{2n+2}y^{4n+4},$

$(S_x + S_y)(M(P_{2n} \vee P_{2n})) = \frac{8}{2n+1}x^{2n+1}y^{2n+1} + (\frac{8n-4}{2n+1} + \frac{4n-2}{n+1})x^{2n+1}y^{2n+2} + \frac{4n^2-7n+4}{n+1}x^{2n+2}y^{2n+2},$

$D_xQ_{-2}J(M(P_{2n} \vee P_{2n})) = 16x^{4n} + (4n+1)(8n-4)x^{4n+1} + (4n+2)(4n^2-7n+4)x^{4n+2},$

$D_xQ_{-2}J(D_x + D_y)(M(P_{2n} \vee P_{2n})) = 32n(2n+1)x^{4n} + (4n+1)(8n-4)(4n+3)x^{4n+1} + (4n+2)(4n + 4)(4n^2-7n+4)x^{4n+2},$

$D_xQ_{-2}J(L_x + L_y)(M(P_{2n} \vee P_{2n})) = \frac{8}{6n+1}x^{6n+1} + \frac{8n-4}{6n+2}x^{6n+2} + \frac{8n-4}{6n+3}x^{6n+3} + \frac{8n^2-14n+8}{6n+4}x^{6n+4},$

$S_xQ_{-2}J(S_x + S_y)(M(P_{2n} \vee P_{2n})) = \frac{8}{8n^2+4n}x^{4n} + \frac{1}{4n+1}\left(\frac{8n-4}{2n+1} + \frac{4n-2}{n+1}\right)x^{4n+1} + \frac{4n^2-7n+4}{(n+1)(4n+2)}x^{4n+2}.$

Now, the results are immediate from these data and Table-1. ■

Theorem 2.3. For $n > 1$, the M-Polynomial of $P_{2n} \vee P_{2n+1}$ is obtained as

$$M(P_{2n} \vee P_{2n+1}; x, y) = 6x^{2n+1}y^{2n+2} + 4(n-1)x^{2n+1}y^{2n+3} + (2^{n-1} + 4n)x^{2n+2}y^{2n+2} + \{2 + 2^{n-1}(2n-1)\}x^{2n+2}y^{2n+3} + (2n-3)x^{2n+3}y^{2n+3}.$$

Proof. Based on the degrees, the set of vertices of $P_{2n} \vee P_{2n+1}$ are partition as:

$V_{2n+1} = \{v \in V(P_{2n} \vee P_{2n+1}): d_v = 2n+1\},$

$V_{2n+2} = \{v \in V(P_{2n} \vee P_{2n+1}): d_v = 2n+2\},$

$V_{2n+3} = \{v \in V(P_{2n} \vee P_{2n+1}): d_v = 2n+3\}.$

From the graph of $P_{2n} \vee P_{2n+1}$, we have

$m_{(2n+1,2n+2)} = 6, m_{(2n+1,2n+3)} = 4(n-1), m_{(2n+2,2n+2)} = 2^{n-1} + 4n, m_{(2n+2,2n+3)} = 2 + 2^{n-1}(2n-1)$ and $m_{(2n+3,2n+3)} = 2n-3$.

Now, by using the definition of M-Polynomial, we have

$M(P_{2n} \vee P_{2n+1}; x, y) = \sum_{\delta \le a \le b \le \Delta} m_{(a,b)}(P_{2n} \vee P_{2n+1}) x^a y^b,$

$= m_{(2n+1,2n+2)} x^{2n+1} y^{2n+2} + m_{(2n+1,2n+3)} x^{2n+1} y^{2n+3} + m_{(2n+2,2n+2)} x^{2n+2} y^{2n+2} + m_{(2n+2,2n+3)} x^{2n+2} y^{2n+3} + m_{(2n+3,2n+3)} x^{2n+3} y^{2n+3},$

$= 6x^{2n+1} y^{2n+2} + 4(n-1) x^{2n+1} y^{2n+3} + \left(2^{n-1} + 4n\right) x^{2n+2} y^{2n+2} + \left\{2 + 2^{n-1}(2n-1)\right\} x^{2n+2} y^{2n+3} + (2n-3) x^{2n+3} y^{2n+3}$. ■

Now, by using the same way as in Theorem-2.2., we have the following result.

Theorem 2.4. The K-Banhatti group of indices of $P_{2n} \vee P_{2n+1}$ are obtained as

1. $B_1(P_{2n} \vee P_{2n+1}) = (2^{n+3} + 2^{n+2} + 120)n^2 + (2^{n+2} + 3.2^n + 8.2^{n-1} + 104)n + (2^{n+1} - 3.2^n - 2^{n-1} - 22)$.
2. $B_2(P_{2n} \vee P_{2n+1}) = (2^{n-4} + 160)n^3 + (2^{n+3} + 6.2^{n-2} + 288)n^2 + (2^{n+3} + 2^{n+2} - 5.2^{n+1} + 9.2^n + 40)n + (2^{n+2} - 15.2^{n-1} - 180)$.
3. $m_{B_1}(P_{2n} \vee P_{2n+1}) = \frac{12}{4n+3} + \frac{2^n + 16n - 8}{4n+4} + \frac{4 + 2^n(2n-1)}{4n+5} + \frac{4n-6}{4n+6}$.
4. $m_{B_2}(P_{2n} \vee P_{2n+1}) = \frac{6}{4n+1}\left(\frac{1}{2n+1} + \frac{1}{2n+2}\right) + \frac{1}{4n+2}\left[4(n-1)\left\{\frac{1}{2n+1} + \frac{1}{2n+3}\right\} + \frac{2^{n-1}+4n}{n-1}\right] + \frac{2 + 2^{n-1}(2n-1)}{4n+3}\left[\frac{1}{2n+2} + \frac{1}{2n+3}\right] + \frac{4n-6}{(2n+3)(4n+4)}$.
5. $HB(P_{2n} \vee P_{2n+1}) = \frac{24}{4n+3} + \frac{2^n + 16n - 8}{2n+2} + \frac{8 + 2^{n+1}(2n-1)}{4n+5} + \frac{4n-6}{2n+3}$.

Theorem 2.5. For $n > 1$, the M-Polynomial of $P_{2n+1} \vee P_{2n+1}$ is obtained as

$$M(P_{2n+1} \vee P_{2n+1}) = 4x^{2n+2} y^{2n+2} + 8n x^{2n+2} y^{2n+3} + (2n-1)^2 x^{2n+3} y^{2n+3}.$$

Proof. Based on the degrees, the set of vertices of $P_{2n+1} \vee P_{2n+1}$ are partition as:

$$V_{2n+2} = \{v \in V(P_{2n+1} \vee P_{2n+1}) : d_v = 2n+2\},$$

$$V_{2n+3} = \{v \in V(P_{2n+1} \vee P_{2n+1}) : d_v = 2n+3\}.$$

From the graph of $P_{2n+1} \vee P_{2n+1}$, we have

$m_{(2n+2,2n+2)} = 4, m_{(2n+2,2n+3)} = 8n$ and $m_{(2n+3,2n+3)} = (2n-1)^2$.

Now, by using the definition of M-Polynomial, we have

$M(P_{2n+1} \vee P_{2n+1}; x, y) = \sum_{\delta \le a \le b \le \Delta} m_{(a,b)}(P_{2n+1} \vee P_{2n+1}) x^a y^b,$

$= m_{(2n+2,2n+2)} x^{2n+2} y^{2n+2} + m_{(2n+2,2n+3)} x^{2n+2} y^{2n+3} + m_{(2n+3,2n+3)} x^{2n+3} y^{2n+3},$

$$= 4x^{2n+2}y^{2n+2} + 8nx^{2n+2}y^{2n+3} + (2n-1)^2x^{2n+3}y^{2n+3}. \blacksquare$$

Now, by using the same way as in Theorem-2.2., we have the following result.

Theorem 2.6. The K-Banhatti group of indices of $P_{2n+1} \vee P_{2n+1}$ are obtained as

1. $B_1(P_{2n+1} \vee P_{2n+1}) = 48n^3 + 104n^2 + 92n + 46.$
2. $B_2(P_{2n+1} \vee P_{2n+1}) = 64n^4 + 224n^3 + 272n^2 + 160n + 56.$
3. $m_{B_1}(P_{2n+1} \vee P_{2n+1}) = \frac{4}{3n+2} + 8n\left[\frac{1}{6n+5} + \frac{1}{6n+6}\right] + \frac{2(2n-1)^2}{6n+7}.$
4. $m_{B_2}(P_{2n+1} \vee P_{2n+1}) = \frac{2}{(n+1)(2n+1)} + \frac{1}{4n+3}\left[\frac{4n}{n+1} + \frac{8n}{2n+3}\right] + \frac{2(2n-1)^2}{(2n+3)(4n+4)}.$
5. $HB(P_{2n+1} \vee P_{2n+1}) = \frac{8}{3n+2} + 16n\left[\frac{1}{6n+5} + \frac{1}{6n+6}\right] + \frac{4(2n-1)^2}{6n+7}.$

2.2. Cartesian Product

Theorem 2.7. For $n > 1$, the M-Polynomial of $P_{2n} \boxdot P_{2n}$ is obtained as

$$M(P_{2n} \boxdot P_{2n}; x, y) = 8x^2y^2 + 4(2n-3)x^3y^3 + 4(2n-2)x^3y^4 + 2(2n-2)(2n-3)x^4y^4.$$

Proof. Based on the degrees, the set of vertices of $P_{2n} \boxdot P_{2n}$ are partition as:

$$V_2 = \{v \in V(P_{2n} \boxdot P_{2n}) : d_v = 2\},$$

$$V_3 = \{v \in V(P_{2n} \boxdot P_{2n}) : d_v = 3\},$$

$$V_4 = \{v \in V(P_{2n} \boxdot P_{2n}) : d_v = 4\}.$$

From the graph of $P_{2n} \boxdot P_{2n}$, we have

$m_{(2,2)} = 8, m_{(3,3)} = 4(2n-3), m_{(3,4)} = 4(2n-2)$ and $m_{(4,4)} = 2(2n-2)(2n-3)$.

Now, by using the definition of M-Polynomial, we have

$$M(P_{2n} \boxdot P_{2n}; x, y) = \sum_{\delta \le a \le b \le \Delta} m_{(a,b)}(P_{2n} \boxdot P_{2n})x^a y^b,$$

$$= m_{(2,2)}x^2y^2 + m_{(3,3)}x^3y^3 + m_{(3,4)}x^3y^4 + m_{(4,4)}x^4y^4,$$

$$= 8x^2y^2 + 4(2n-3)x^3y^3 + 4(2n-2)x^3y^4 + 2(2n-2)(2n-3)x^4y^4. \blacksquare$$

Now, by using the same way as in Theorem-2.2., we have the following result.

Theorem 2.8. The K-Banhatti group of indices of $P_{2n} \boxdot P_{2n}$ are obtained as

1. $B_1(P_{2n} \boxdot P_{2n}) = 160n^2 - 152n.$
2. $B_2(P_{2n} \boxdot P_{2n}) = 384n^2 - 488n + 72.$
3. $m_{B_1}(P_{2n} \boxdot P_{2n}) = -\frac{8}{5}n^2 + \frac{11}{63}n + \frac{341}{315}.$
4. $m_{B_2}(P_{2n} \boxdot P_{2n}) = \frac{2}{3}n^2 + \frac{3}{5}n + \frac{31}{15}.$
5. $HB(P_{2n} \boxdot P_{2n}) = -\frac{16}{5}n^2 + \frac{22}{63}n + \frac{682}{315}.$

Theorem 2.9. For $n > 1$, the M-Polynomial of $P_{2n} \boxdot P_{2n+1}$ is obtained as

$M(P_{2n} \boxdot P_{2n+1}; x, y) = 8x^2y^3 + (8n-10)x^3y^3 + (8n-6)x^3y^4 + \{(2n-2)^2 + (2n-1)(2n-3)\}x^4y^4$.

Proof. Based on the degrees, the set of vertices of $P_{2n} \boxdot P_{2n+1}$ are partition as:

$$V_2 = \{v \in V(P_{2n} \boxdot P_{2n+1}): d_v = 2\},$$

$$V_3 = \{v \in V(P_{2n} \boxdot P_{2n+1}): d_v = 3\},$$

$$V_4 = \{v \in V(P_{2n} \boxdot P_{2n+1}): d_v = 4\}.$$

From the graph of $P_{2n} \boxdot P_{2n+1}$, we have

$m_{(2,3)} = 8, m_{(3,3)} = 8n - 10, m_{(3,4)} = 8n - 6$ and $m_{(4,4)} = \{(2n-2)^2 + (2n-1)(2n-3)\}$.

Now, by using the definition of M-Polynomial, we have

$$M(P_{2n} \boxdot P_{2n+1}; x, y) = \sum_{\delta \le a \le b \le \Delta} m_{(a,b)}(P_{2n} \boxdot P_{2n+1}) x^a y^b,$$

$$= m_{(2,3)}x^2y^3 + m_{(3,3)}x^3y^3 + m_{(3,4)}x^3y^4 + m_{(4,4)}x^4y^4,$$

$$= 8x^2y^3 + (8n-10)x^3y^3 + (8n-6)x^3y^4 + \{(2n-2)^2 + (2n-1)(2n-3)\}x^4y^4. \blacksquare$$

Now, by using the same way as in Theorem-2.2., we have the following result.

Theorem 2.10. The K-Banhatti group of indices of $P_{2n} \boxdot P_{2n+1}$ are obtained as

1. $B_1(P_{2n} \boxdot P_{2n+1}) = 160n^2 - 72n - 14$.
2. $B_2(P_{2n} \boxdot P_{2n+1}) = 384n^2 - 296n + 6$.
3. $m_{B_1}(P_{2n} \boxdot P_{2n+1}) = -\frac{8}{5}n^2 + \frac{2323}{315}n - \frac{1151}{420}$.
4. $m_{B_2}(P_{2n} \boxdot P_{2n+1}) = \frac{2}{3}n^2 + \frac{14}{15}n - \frac{79}{180}$.
5. $HB(P_{2n} \boxdot P_{2n+1}) = -\frac{16}{5}n^2 + \frac{4646}{315}n - \frac{1151}{210}$.

Theorem 2. 11. For $n > 1$, the M-Polynomial of $P_{2n+1} \boxdot P_{2n+1}$ is obtained as

$$M(P_{2n+1} \boxdot P_{2n+1}; x, y) = 8x^2y^3 + (8n-8)x^3y^3 + (8n-4)x^3y^4 + 2(2n-2)(2n-1)x^4y^4.$$

Proof. Based on the degrees, the set of vertices of $P_{2n+1} \boxdot P_{2n+1}$ are partition as:

$$V_2 = \{v \in V(P_{2n+1} \boxdot P_{2n+1}): d_v = 2\},$$

$$V_3 = \{v \in V(P_{2n+1} \boxdot P_{2n+1}): d_v = 3\},$$

$$V_4 = \{v \in V(P_{2n+1} \boxdot P_{2n+1}): d_v = 4\}.$$

From the graph of $P_{2n+1} \boxdot P_{2n+1}$, we have

$m_{(2,3)} = 8, m_{(3,3)} = 8n - 8, m_{(3,4)} = 8n - 4$ and $m_{(4,4)} = 2(2n-2)(2n-1)$.

Now, by using the definition of M-Polynomial, we have

$$M(P_{2n+1} \boxdot P_{2n+1}; x, y) = \sum_{\delta \le a \le b \le \Delta} m_{(a,b)}(P_{2n+1} \boxdot P_{2n+1}) x^a y^b,$$

$$= m_{(2,3)} x^2 y^3 + m_{(3,3)} x^3 y^3 + m_{(3,4)} x^3 y^4 + m_{(4,4)} x^4 y^4,$$

$$= 8x^2 y^3 + (8n - 8)x^3 y^3 + (8n - 4)x^3 y^4 + 2(2n - 2)(2n - 1)x^4 y^4. \blacksquare$$

Now, by using the same way as in Theorem-2.2., we have the following result.

Theorem 2.12. The K-Banhatti group of indices of $P_{2n+1} \boxdot P_{2n+1}$ are obtained as

1. $B_1(P_{2n+1} \boxdot P_{2n+1}) = 160n^2 + 108n - 12.$
2. $B_2(P_{2n+1} \boxdot P_{2n+1}) = 384n^2 - 104n - 20.$
3. $m_{B_1}(P_{2n+1} \boxdot P_{2n+1}) = \frac{8}{5}n^2 + \frac{559}{315}n + \frac{317}{630}.$
4. $m_{B_2}(P_{2n+1} \boxdot P_{2n+1}) = \frac{2}{3}n^2 + \frac{19}{15}n + \frac{34}{45}.$
5. $HB(P_{2n+1} \boxdot P_{2n+1}) = \frac{16}{5}n^2 + \frac{1118}{315}n + \frac{317}{315}.$

2.3. Lexicographic Product

Theorem 2.13. For $n > 1$, the M-polynomial of $P_{2n} \circ P_{2n}$ is obtained as

$$M(P_{2n} \circ P_{2n}; x, y) = 8x^3 y^5 + 4x^3 y^8 + (8n - 8)x^5 y^5 + (24n - 32)x^5 y^8 + 2(2n - 3)(4n - 5)x^8 y^8.$$

Proof. Based on the degrees, the set of vertices of $P_{2n} \circ P_{2n}$ are partition as:

$$V_3 = \{v \in V(P_{2n} \circ P_{2n}): d_v = 3\},$$

$$V_5 = \{v \in V(P_{2n} \circ P_{2n}): d_v = 5\},$$

$$V_8 = \{v \in V(P_{2n} \circ P_{2n}): d_v = 8\}.$$

From the graph of $P_{2n} \circ P_{2n}$, we have

$m_{(3,5)} = 8, m_{(3,8)} = 4, m_{(5,5)} = (8n - 8), m_{(5,8)} = (24n - 32), and\ m_{(8,8)} = 2(2n - 3)(4n - 5).$

Now, by using the definition of M-Polynomial, we have

$$M(P_{2n} \circ P_{2n}; x, y) = \sum_{\delta \le a \le b \le \Delta} m_{(a,b)}(P_{2n} \circ P_{2n}) x^a y^b,$$

$$= m_{(3,5)} x^3 y^5 + m_{(3,8)} x^3 y^8 + m_{(5,5)} x^5 y^5 + m_{(5,8)} x^5 y^8 + m_{(8,8)} x^8 y^8,$$

$$= 8x^3 y^5 + 4x^3 y^8 + (8n - 8)x^5 y^5 + (24n - 32)x^5 y^8 + 2(2n - 3)(4n - 5)x^8 y^8. \blacksquare$$

Now, by using the same way as in Theorem 2.2., we have the following result.

Theorem 2.14. The K-Banhatti group of indices of $P_{2n} \circ P_{2n}$ are obtained as

1. $B_1(P_{2n} \circ P_{2n}) = 704n^2 - 888n + 268$
2. $B_2(P_{2n} \circ P_{2n}) = 3584n^2 - 5784n + 2284.$

3. $m_{B_1}(P_{2n} \circ P_{2n}) = \frac{16}{11}n^2 - \frac{3}{494}n - \frac{5825}{24453}$.
4. $m_{B_2}(P_{2n} \circ P_{2n}) = \frac{2}{7}n^2 + \frac{249}{770}n + \frac{4369}{41580}$.
5. $HB(P_{2n} \circ P_{2n}) = \frac{32}{11}n^2 - \frac{3}{247}n - \frac{11650}{24453}$.

Theorem 2.15. For $n > 1$, the M-polynomial of $P_{2n} \circ P_{2n+1}$ is obtained as

$$M(P_{2n} \circ P_{2n+1}; x, y) = 8x^3y^5 + 4x^3y^8 + (8n-6)x^5y^5 + (24n-26)x^5y^8 + \{(2n-2)^2 + (2n-3)(6n-5)\}x^8y^8.$$

Proof. Based on the degrees, the set of vertices of $P_{2n} \circ P_{2n+1}$ are partition as:

$$V_3 = \{v \in V(P_{2n} \circ P_{2n+1}) : d_v = 3\},$$

$$V_5 = \{v \in V(P_{2n} \circ P_{2n+1}) : d_v = 5\},$$

$$V_8 = \{v \in V(P_{2n} \circ P_{2n+1}) : d_v = 8\}.$$

From the graph of $P_{2n} \circ P_{2n+1}$, we have

$m_{(3,5)} = 8, m_{(3,8)} = 4, m_{(5,5)} = (8n-6), m_{(5,8)} = (24n-26), and\ m_{(8,8)} = \{(2n-2)^2 + (2n-3)(6n-5)\}$.

Now, by using the definition of M-polynomial, we have

$$M(P_{2n} \circ P_{2n+1}; x, y) = \sum_{\delta \le a \le b \le \Delta} m_{(a,b)}(P_{2n} \circ P_{2n+1})x^a y^b,$$

$$= m_{(3,5)}x^3y^5 + m_{(3,8)}x^3y^8 + m_{(5,5)}x^5y^5 + m_{(5,8)}x^5y^8 + m_{(8,8)}x^8y^8,$$

$$= 8x^3y^5 + 4x^3y^8 + (8n-6)x^5y^5 + (24n-26)x^5y^8 + \{(2n-2)^2 + (2n-3)(6n-5)\}x^8y^8.$$ ■

Now, by using the same way as in Theorem 2.2., we have the following result.

Theorem 2.16. The K-Banhatti group of indices of $P_{2n} \circ P_{2n+1}$ are obtained as

1. $B_1(P_{2n} \circ P_{2n+1}) = 704n^2 - 536n + 46$.
2. $B_2(P_{2n} \circ P_{2n+1}) = 3584n^2 - 3992n + 838$.
3. $m_{B_1}(P_{2n} \circ P_{2n+1}) = \frac{16}{11}n^2 - \frac{3919}{5434}n - \frac{46897}{195624}$.
4. $m_{B_2}(P_{2n} \circ P_{2n+1}) = \frac{2}{7}n^2 - \frac{61}{770}n + \frac{15461}{83160}$.
5. $HB(P_{2n} \circ P_{2n+1}) = \frac{32}{11}n^2 + \frac{3919}{2717}n - \frac{46797}{97812}$.

Theorem 2.17. For $n > 1$, the M-polynomial of $P_{2n+1} \circ P_{2n+1}$ is obtained as

$$M(P_{2n+1} \circ P_{2n+1}; x, y) = 8x^3y^5 + 4x^3y^8 + (8n-4)x^5y^5 + (24n-20)x^5y^8 + 2(2n-2)(4n-3)x^8y^8.$$

Proof. Based on the degrees, the set of vertices of $P_{2n+1} \circ P_{2n+1}$ are partition as:

$$V_3 = \{v \in V(P_{2n+1} \circ P_{2n+1}) : d_v = 3\},$$

$$V_5 = \{v \in V(P_{2n+1} \circ P_{2n+1}): d_v = 5\},$$

$$V_8 = \{v \in V(P_{2n+1} \circ P_{2n+1}): d_v = 8\}.$$

From the graph of $P_{2n+1} \circ P_{2n+1}$, we have

$m_{(3,5)} = 8, m_{(3,8)} = 4, m_{(5,5)} = (8n-4), m_{(5,8)} = (24n-20), and\ m_{(8,8)} = 2(2n-2)(4n-3).$

Now, by using the definition of M-polynomial, we have

$M(P_{2n+1} \circ P_{2n+1}; x, y) = \sum_{\delta \le a \le b \le \Delta} m_{(a,b)}(P_{2n+1} \circ P_{2n+1}) x^a y^b,$

$= m_{(3,5)} x^3 y^5 + m_{(3,8)} x^3 y^8 + m_{(5,5)} x^5 y^5 + m_{(5,8)} x^5 y^8 + m_{(8,8)} x^8 y^8,$

$= 8x^3y^5 + 4x^3y^8 + (8n-4)x^5y^5 + (24n-20)x^5y^8 + 2(2n-2)(4n-3)x^8y^8.$ ■

Now, by using the same way as in Theorem 2.2., we have the following result.

Theorem 2.18. The K-Banhatti group of indices of $P_{2n+1} \circ P_{2n+1}$ are obtained as

1. $B_1(P_{2n+1} \circ P_{2n+1}) = 704n^2 - 184n + 16.$
2. $B_2(P_{2n+1} \circ P_{2n+1}) = 3584n^2 - 2200n + 288.$
3. $m_{B_1}(P_{2n+1} \circ P_{2n+1}) = \frac{16}{11}n^2 + \frac{7871}{5434}n + \frac{594755}{1662804}.$
4. $m_{B_2}(P_{2n+1} \circ P_{2n+1}) = \frac{2}{7}n^2 + \frac{67}{110}n + \frac{7031}{20790}.$
5. $HB(P_{2n+1} \circ P_{2n+1}) = \frac{32}{11}n^2 + \frac{7871}{2717}n + \frac{594755}{831402}.$

2.4. Corona Product

Theorem 2.19. For $n > 1$, the M-polynomial of $P_{2n} \odot P_{2n}$ is obtained as

$$M(P_{2n} \odot P_{2n}; x, y) = 4nx^2y^3 + 4x^2y^{2n+1} + 2(2n-2)x^2y^{2n+2} + 2n(2n-3)x^3y^3 + 2(2n-2)x^3y^{2n+1} + (2n-2)^2x^3y^{2n+2} + 2x^{2n+1}y^{2n+2} + (2n-3)x^{2n+2}y^{2n+2}.$$

Proof. Based on the degrees, the set of vertices of $P_{2n} \odot P_{2n}$ are partition as:

$$V_2 = \{v \in V(P_{2n} \odot P_{2n}): d_v = 2\},$$

$$V_3 = \{v \in V(P_{2n} \odot P_{2n}): d_v = 3\},$$

$$V_{2n+1} = \{v \in V(P_{2n} \odot P_{2n}): d_v = 2n+1\},$$

$$V_{2n+2} = \{v \in V(P_{2n} \odot P_{2n}): d_v = 2n+2\}.$$

From the graph of $P_{2n} \odot P_{2n}$, we have

$m_{(2,3)} = 4n, m_{(2,2n+1)} = 4, m_{(2,2n+2)} = 2(2n-2), m_{(3,3)} = 2n(2n-3), m_{(3,2n+1)} = 2(2n-2), m_{(3,2n+2)} = (2n-2)^2, m_{(2n+1,2n+2)} = 2\ and\ m_{(2n+2,2n+2)} = (2n-3).$

Now, by using the definition of M-polynomial, we have

$M(p_{2n}\odot p_{2n};x,y)=\sum_{\delta\le a\le b\le\Delta} m_{(a,b)}(P_{2n}\odot P_{2n})x^a y^b,$

$= m_{(2,3)}x^2y^3 + m_{(2,2n+1)}x^2y^{2n+1} + m_{(2,2n+2)}x^2y^{2n+2} + m_{(3,3)}x^3y^3 + m_{(3,2n+1)}x^3y^{2n+1} + m_{(3,2n+2)}x^3y^{2n+2} + m_{(2n+1,2n+2)}x^{2n+1}y^{2n+2} + m_{(2n+2,2n+2)}x^{2n+2}y^{2n+2},$

$= 4nx^2y^3 + 4x^2y^{2n+1} + 2(2n-2)x^2y^{2n+2} + 2n(2n-3)x^3y^3 + 2(2n-2)x^3y^{2n+1} + (2n-2)^2x^3y^{2n+2} + 2x^{2n+1}y^{2n+2} + (2n-3)x^{2n+2}y^{2n+2}$. ■

Now by using the same way as in Theorem 2.2., we have the following result.

Theorem 2.20. The K-Banhatti group of indices of $p_{2n}\odot p_{2n}$ are obtained as

1. $B_1(P_{2n}\odot P_{2n}) = 24n^3 + 92n^2 - 32n - 2.$
2. $B_2(P_{2n}\odot P_{2n}) = 16n^4 + 80n^3 + 28n^2 - 4n + 22.$
3. $m_{B_1}(P_{2n}\odot P_{2n}) = -\frac{91}{12}+\frac{289n}{105}+\frac{8n^2}{7}-\frac{4}{1+n}+\frac{32}{3+n}+\frac{4}{3+2n}-\frac{16}{5+2n}+\frac{1}{1+3n}-\frac{9}{3+4n}+\frac{81}{4(5+4n)}+\frac{8}{3+6n}-\frac{13}{6+9n}.$
4. $m_{B_2}(P_{2n}\odot P_{2n}) = \frac{7}{9}n+\frac{2}{3}n^2+\frac{47+n\left(175+4n\left(45+4n\left(19+n\left(27+n(13+4n)\right)\right)\right)\right)}{6(1+n)^3(1+2n)^2(3+2n)(1+4n)}.$
5. $HB(P_{2n}\odot P_{2n}) = -\frac{91}{6}+\frac{578}{105}n+\frac{16}{7}n^2-\frac{8}{1+n}+\frac{64}{3+n}+\frac{8}{3+2n}-\frac{32}{5+2n}-\frac{2}{1+3n}-\frac{81}{2(5+4n)}+\frac{16}{3+6n}-\frac{26}{6+9n}.$

Theorem 2.21. For $n>1$, the M-polynomial of of $p_{2n}\odot p_{2n+1}$ is obtain as

$M(P_{2n}\odot P_{2n+1};x,y) = 4nx^2y^3 + 4x^2y^{2n+2} + 2(2n-2)x^2y^{2n+3} + 2n(2n-2)x^3y^3 + 2n(2n-1)x^3y^{2n+2} + (2n-1)(2n-2)x^3y^{2n+3} + 2x^{2n+2}y^{2n+3} + (2n-3)x^{2n+3}y^{2n+3}.$

Proof. Based on the degrees, the set of vertices of $P_{2n}\odot P_{2n+1}$ are partition as:

$$V_2 = \{v\in V(P_{2n}\odot P_{2n+1}): d_v = 2\},$$

$$V_3 = \{v\in V(P_{2n}\odot P_{2n+1}): d_v = 3\},$$

$$V_{2n+2} = \{v\in V(P_{2n}\odot P_{2n+1}): d_v = 2n+2\},$$

$$V_{2n+3} = \{v\in V(P_{2n}\odot P_{2n+1}): d_v = 2n+3\}.$$

From the graph of $P_{2n}\odot P_{2n+1}$, we have

$m_{(2,3)} = 4n, m_{(2,2n+2)} = 4, m_{(2,2n+3)} = 2(2n-2), m_{(3,3)} = 2n(2n-2), m_{(3,2n+2)} = 2n(2n-1), m_{(3,2n+3)} = (2n-1)(2n-2), m_{(2n+2,2n+3)} = 2\ and\ m_{(2n+3,2n+3)} = (2n-3).$

Now, by using the definition of M-polynomial, we have

$M(P_{2n}\odot P_{2n+1};x,y)=\sum_{\delta\le a\le b\le\Delta} m_{(a,b)}(P_{2n}\odot P_{2n+1})x^a y^b,$

$= m_{(2,3)}x^2y^3 + m_{(2,2n+2)}x^2y^{2n+2} + m_{(2,2n+3)}x^2y^{2n+3} + m_{(3,3)}x^3y^3 + m_{(3,2n+2)}x^3y^{2n+2}+, m_{(3,2n+3)}x^3y^{2n+3} + m_{(2n+2,2n+3)}x^{2n+2}y^{2n+3} + m_{(2n+3,2n+3)}x^{2n+3}y^{2n+3},$

$= 4nx^2y^3 + 4x^2y^{2n+2} + 2(2n-2)x^2y^{2n+3} + 2n(2n-2)x^3y^3 + 2n(2n-1)x^3y^{2n+2} + (2n-1)(2n-2)x^3y^{2n+3} + 2x^{2n+2}y^{2n+3} + (2n-3)x^{2n+3}y^{2n+3}$. ■

Now, by using the same way as in Theorem 2.2., we have the following result.

Theorem 2.22. The K-Banhatti group of indices of $p_{2n}\odot p_{2n+1}$ are obtained as

1. $B_1(P_{2n}\odot P_{2n+1}) = 64n^3 + 224n^2 - 48n + 28$.
2. $B_2(P_{2n}\odot P_{2n+1}) = 64n^4 + 208n^3 + 188n^2 - 210n - 28$.
3. $m_{B_1}(P_{2n}\odot P_{2n+1}) = -\frac{55}{3} + \frac{4}{3(1+n)} + \frac{2}{2+n} + \frac{21}{3+n} - \frac{4}{3+2n} - \frac{12}{5+2n} + \frac{72}{7+2n} + \frac{35}{4(5+4n)} + \frac{99}{4(7+4n)} + \frac{2}{5+6n} - \frac{32}{3(7+6n)} + \frac{8}{105}n(83+15n)$.
4. $m_{B_2}(P_{2n}\odot P_{2n+1}) = \frac{1}{18}\left(4n(8+3n) + 3\left(-4 - \frac{3}{1+n} - \frac{60}{2+n} - \frac{48}{(3+2n)^2} + \frac{92}{3+2n} + \frac{32}{3+4n}\right)\right)$.
5. $HB(P_{2n}\odot P_{2n+1}) = -\frac{110}{3} + \frac{8}{3(1+n)} + \frac{4}{2+n} + \frac{42}{3+n} + \frac{64}{3+n} - \frac{8}{3+2n} - \frac{24}{5+2n} + \frac{144}{7+2n} + \frac{4}{5+6n} - \frac{64}{3(7+6n)} + \frac{35}{10+8n} + \frac{99}{14+8n} + \frac{16}{105}n(83+15n)$.

Theorem 2.23. For $n > 1$, the M-polynomial of of $p_{2n+1}\odot p_{2n+1}$ is obtained as

$M(P_{2n+1}\odot P_{2n+1}; x, y) = 2(2n+1)x^2y^3 + 4x^2y^{2n+2} + 2(2n-1)x^2y^{2n+3} + (2n+1)(2n-2)x^3y^3 + 2(2n-1)x^3y^{2n+2} + (2n-1)^2x^3y^{2n+3} + 2x^{2n+2}y^{2n+3} + (2n-2)x^{2n+3}y^{2n+3}$.

Proof. Based on the degrees, the set of vertices of $P_{2n+1}\odot P_{2n+1}$are partition as:

$$V_2 = \{v \in V(P_{2n+1}\odot P_{2n+1}): d_v = 2\},$$

$$V_3 = \{v \in V(P_{2n+1}\odot P_{2n+1}): d_v = 3\},$$

$$V_{2n+2} = \{v \in V(P_{2n+1}\odot P_{2n+1}): d_v = 2n+2\},$$

$$V_{2n+3} = \{v \in V(P_{2n+1}\odot P_{2n+1}): d_v = 2n+3\}.$$

From the graph of $P_{2n}\odot P_{2n+1}$, we have

$m_{(2,3)} = 2(2n+1), m_{(2,2n+2)} = 4, m_{(2,2n+3)} = 2(2n-1), m_{(3,3)} = (2n+1)(2n-2), m_{(3,2n+2)} = 2(2n-1), m_{(3,2n+3)} = (2n-1)^2, m_{(2n+2,2n+3)} = 2\ and\ m_{(2n+3,2n+3)} = (2n-2)$.

Now, by using the definition of M-polynomial, we have

$M(P_{2n+1}\odot P_{2n+1}; x, y) = \sum_{\delta\le a\le b\le\Delta} m_{(a,b)}(P_{2n+1}\odot P_{2n+1})x^ay^b$,

$= m_{(2,3)}x^2y^3 + m_{(2,2n+2)}x^2y^{2n+2}+, m_{(2,2n+3)}x^2y^{2n+3} + m_{(3,3)}x^3y^3 + m_{(3,2n+2)}x^3y^{2n+2} + m_{(3,2n+3)}x^3y^{2n+3} + m_{(2n+2,2n+3)}x^{2n+2}y^{2n+3} + m_{(2n+3,2n+3)}x^{2n+3}y^{2n+3}$,

$= 2(2n+1)x^2y^3 + 4x^2y^{2n+2} + 2(2n-1)x^2y^{2n+3} + (2n+1)(2n-2)x^3y^3 + 2(2n-1)x^3y^{2n+2} + (2n-1)^2x^3y^{2n+3} + 2x^{2n+2}y^{2n+3} + (2n-2)x^{2n+3}y^{2n+3}$. ■

Now, by using the same way as in Theorem 2.2., we have the following result.

Theorem 2.24. The K-Banhatti group of indices of $p_{2n+1}\odot p_{2n+1}$ are obtained as

1. $B_1(p_{2n+1}\odot p_{2n+1}) = 24n^3 + 156n^2 + 86n - 10.$
2. $B_2(p_{2n+1}\odot p_{2n+1}) = 16n^4 + 128n^3 + 324n^2 + 72n - 40.$
3. $m_{B_1}(p_{2n+1}\odot p_{2n+1}) = -\frac{2683}{420} + \frac{409n}{105} + \frac{8n^2}{7} + \frac{4}{3(1+n)} + \frac{2}{n+2} - \frac{7}{n+3} - \frac{4}{3+2n} - \frac{12}{5+2n} + \frac{64}{2n+7} - \frac{7}{5+4n} + \frac{81}{4(7+4n)} + \frac{2}{5+6n} - \frac{26}{3(7+6n)}.$
4. $m_{B_2}(p_{2n+1}\odot p_{2n+1}) = \frac{1}{9}\left(8 + \frac{9}{(n+1)^2}. - \frac{18}{n+1} - \frac{75}{n+2} - \frac{144}{(2n+3)^2} + \frac{159}{2n+3} + \frac{48}{4n+3} + n(13 + 6n)\right).$
5. $HB(p_{2n+1}\odot p_{2n+1}) = -\frac{2683}{210} + \frac{818n}{105} + \frac{16n^2}{7} + \frac{8}{3(n+1)} + \frac{4}{n+2} - \frac{14}{n+3} - \frac{8}{2n+3} - \frac{24}{2n+5} + \frac{128}{2n+7} - \frac{14}{4n+5} + \frac{81}{2(4n+7)} + \frac{4}{6n+5} - \frac{52}{3(6n+7)}.$

3. Some Graphical Comparison between the Computing M-Polynomials and K-Banhatti Indices

3.1. Graphical Comparison of Computed M-Polynomials

The three-dimensional graphical representations of our computing M-Polynomials are shown in Figure 1-4.

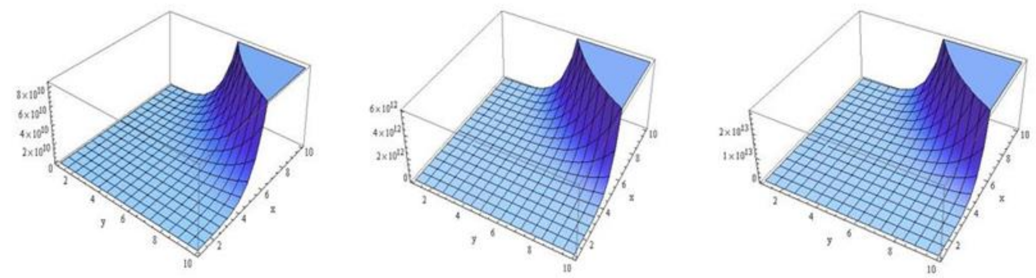

Figure 1: M-Polynomial of $P_{2n}\vee P_{2n}, P_{2n}\vee P_{2n+1}$ and $P_{2n+1}\vee P_{2n+1}$ respectively

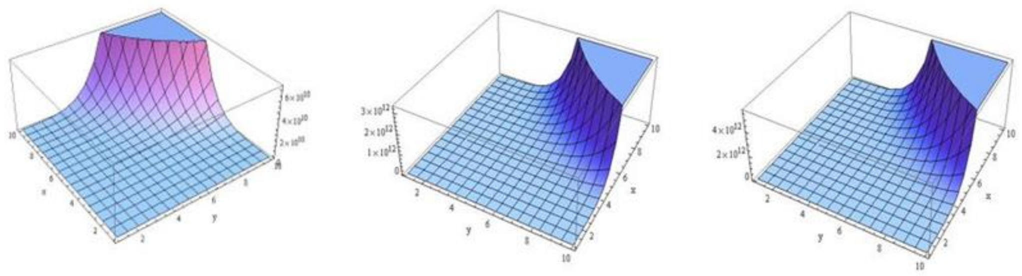

Figure 2: M-Polynomial of $P_{2n} \boxdot P_{2n}, P_{2n} \boxdot P_{2n+1}$ and $P_{2n+1} \boxdot P_{2n+1}$ respectively

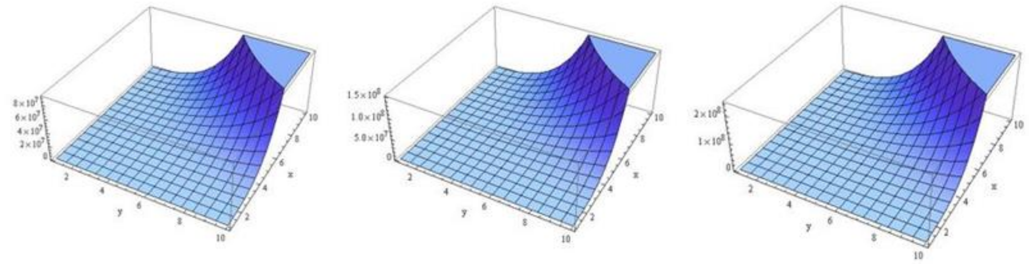

Figure 3: M-Polynomial of $P_{2n} \circ P_{2n}, P_{2n} \circ P_{2n+1}$ and $P_{2n+1} \circ P_{2n+1}$ respectively

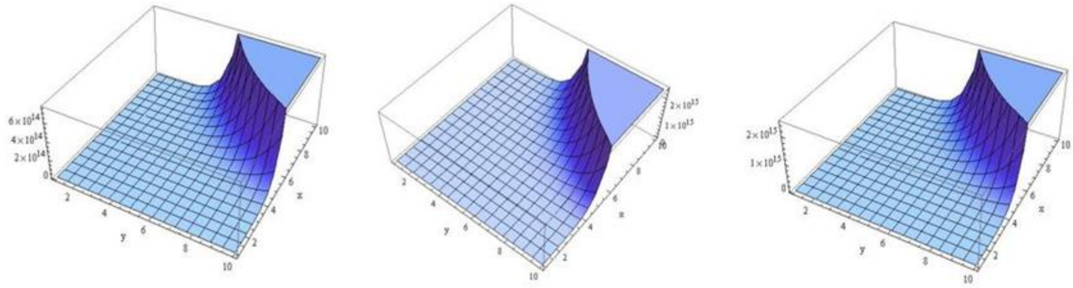

Figure 4: M-Polynomial of $P_{2n} \odot P_{2n}, P_{2n} \odot P_{2n+1}$ and $P_{2n+1} \odot P_{2n+1}$ respectively

The M-polynomial holds a wealth of knowledge on degree-based topological indices. We expect that a more in depth look at the characteristics of M-polynomials will lead to new general insights in the study of topological indices. Mathematica is used to visualise the surface plotting of polynomials. To draw these graphs, we first use the x and y parameters to form a horizontal grid and then we build a surface on the top of it. These graphs show how the polynomials behave differently depending on the parameters. We observed that all the computing M-Polynomials grows with respect to the values of the parameters x and y. But among these computing M-Polynomials of different graph operations, M-Polynomials of $P_{2n} \odot P_{2n+1}$ and $P_{2n+1} \odot P_{2n+1}$ are grows fast while the M-Polynomial of $P_{2n} \circ P_{2n}$ grows slowly.

3.2. Some Graphical Comparison of Computed K-Banhatti Indices

The graphical representations of our computing Banhatti indices are shown in Figure 5-9.

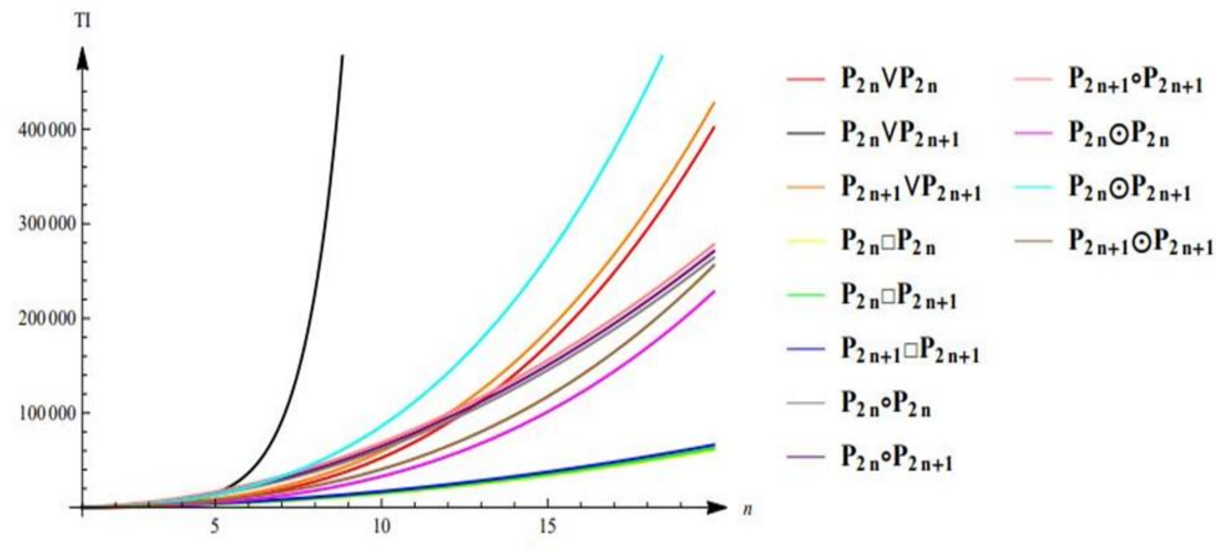

Figure 5: Comparison of First K-Banhatti index among the considering graph operations

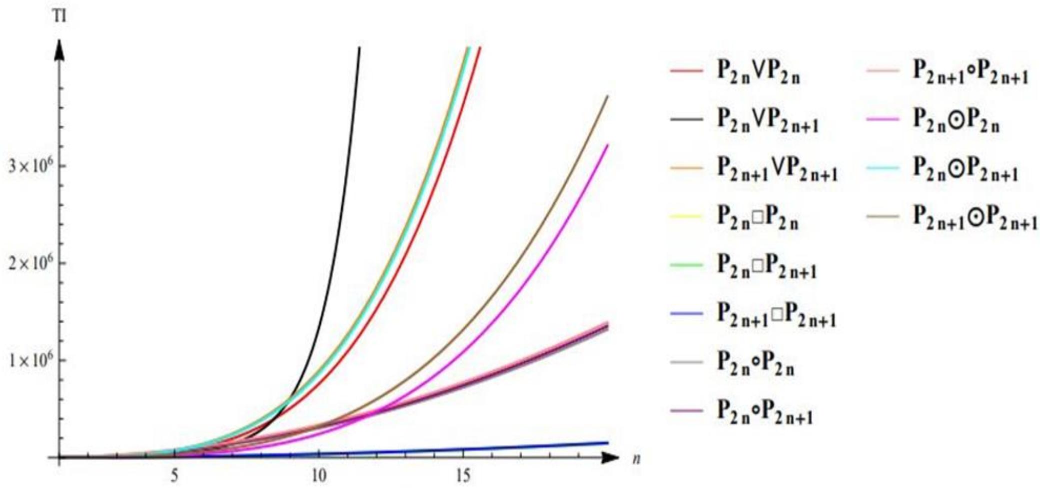

Figure 6: Comparison of Second K-Banhatti index among the considering graph operations

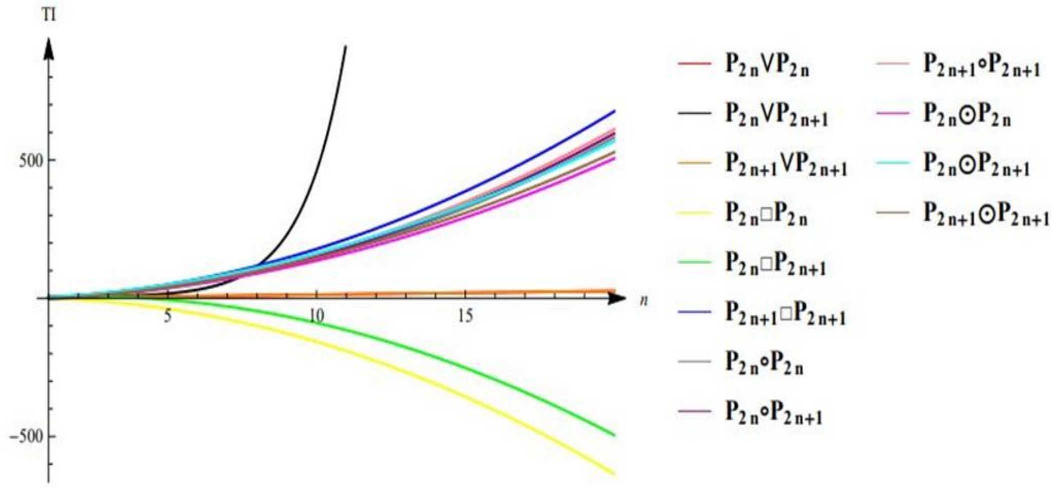

Figure 7: Comparison of Modified first K-Banhatti index among the considering graph operatins

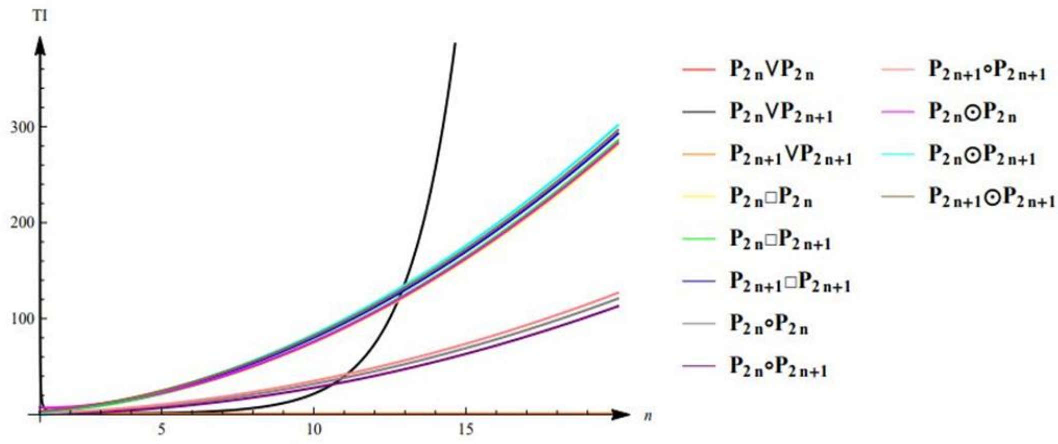

Figure 8: Comparison of Modified second K-Banhatti index among the considering graph operations

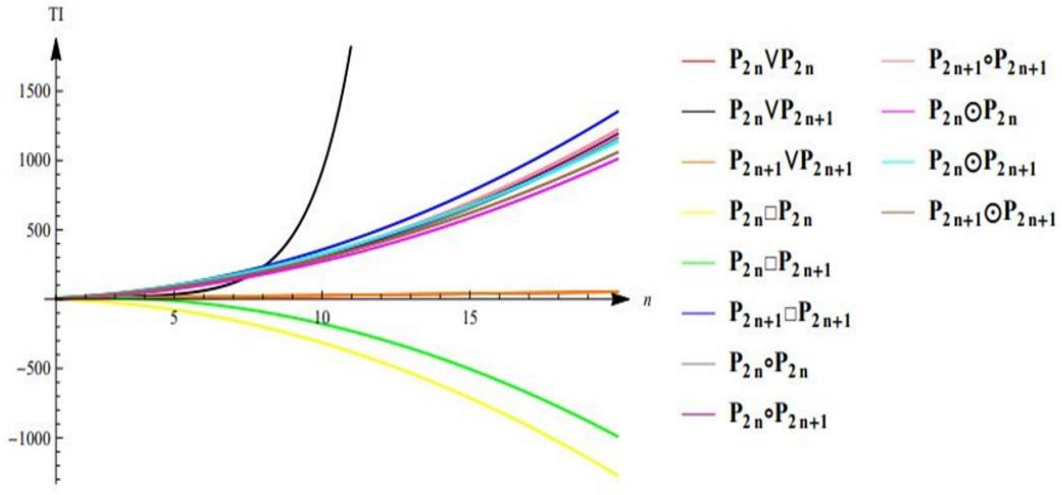

Figure 9: Comparison of Harmonic K-Banhatti index among the considering graph operations

We are using Mathematica to visualise these two-dimensional graphs. Through these graphs, we are trying to find some comparison among the considering Banhatti group of indices of some graph operations on paths of different orders. There is a common thing in all the graphs that all the computing indices grows fast for the operation $P_{2n} \vee P_{2n}$ as compared to other operations. These comparisons should be helpful for finding some hidden information of the considering huge graph structures.

4. Conclusion

The main objective of this article is to evaluate the M-Polynomials of some graph operations. Basically, we are considering some graph operations on paths of different degrees. We first compute the M-Polynomials of the considered graph operations and then we evaluate the K-Banhatti group of indices using these M-Polynomials. Further, we also present some graphical comparison among the M-Polynomials and also the K-Banhatti group of indices for the graph operations.

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