Strong Weak Secure Domination in Graphs

J. Annaal Mercy[#1], L. Benedict Michael Raj [*2], K. A Germina [#3]
[#1,*2]PG and Research Department of Mathematics, St. Joseph's College, Trichy 620 002, India.
[#3] Department of Mathematics, Central University of Kerala, Kasargod, Kerala, India.
annaalmercy@gmail.com, benedict.mraj@gmail.com, germinaka@cukerala.ac.in

Issue: *Special Issue on Mathematical Computation in Combinatorics and Graph Theory in Mathematical Statistician and Engineering Applications*

Article Info
Page Number: 56 - 62
Publication Issue:
Vol 71 No. 3s3 (2022)

Article Received: *30 April 2022*
Revised: *22 May 2022*
Accepted: *25 June 2022*
Publication: *02 August 2022*

Abstract
Let G be a graph. A subset X of V is a Secure Dominating Set(SDS)[5] if for every x in $V - X$, there exists some y in X adjacent to x such that $(X - \{y\}) \cup \{x\}$ is a dominating set. A SDS X of V is called a Strong Secure Dominating Set(SSDS) if for every x in $V - X$, there exists some y in X such that $d(y) \geq d(x)$. Similarly, Weak Secure Dominating Set(WSDS) is defined. The minimum cardinality of a strong(weak) secure dominating set is denoted by $\gamma_{ss}(G)$($\gamma_{ws}(G)$). We initiate a study on these parameters and some bounds related to them are obtained.

Keywords:Domination; secure Domination; strong domination; weak domination; strong secure domination; weak secure domination.
2010 Mathematics Subject Classification: 05C69

1. Introduction

Throughout this article, unless otherwise mentioned, by a graph we mean a connected, simple graph and any terms which are not mentioned here, the reader may refer to [2], [3] [4] and [7-10]. A set $X \subseteq V$ is called a **dominating set** of G, if for every vertex x in $V - X$, there exists at least one y in X such that x and y are adjacent. A dominating set X is called a strong (weak, respectively) dominating set if for every x in $V - X$, there exists some y in X such that $d(y) \geq d(x)(d(x) \geq d(y)$, respectively). The minimum cardinality of a strong (weak, respectively) dominating set is called a **strong (weak, respectively) domination number** and is denoted by $\gamma_{st}(G)$ ($\gamma_w(G)$, respectively). The concept of strong and weak domination was introduced by Sampath Kumar and Pushpa Latha [6]. A dominating set X is called a **secure dominating set** of G if for every x in $V - X$, there exists some y in X adjacent to x such that $(X - \{y\}) \cup \{x\}$ is a dominating set. The minimum cardinality of a secure dominating set is called secure domination number and it is denoted by $\gamma_s(G)$. This concept of secure domination was introduced by E.J.Cockayne [1]. They also define a vertex v to be an X-external private neighbour of w if $N(w) \cap X = \{v\}$ for any v in X and w in $V - X$ and is denoted by $P_n(v, X)$.

In this paper we define strong secure dominating set denoted as SSDS. A secure dominating set X of graph G is called a **strong secure dominating set** if for every x in $V - X$, there exists some y in X such that x and y are adjacent and $d(y) \geq d(x)$; and the minimum cardinality of a SSDS is called a strong secure domination number and it is denoted respectively as $\gamma_{ss}(G)$.

Similarly we define weak secure dominating set denoted as WSDS. A secure dominating set X of graph G is called a **Weak secure dominating set** if for every x in $V - X$, there exists some y in X such that x and y are adjacent and $d(y) \leq d(x)$; and the minimum cardinality of a WSDS is called a weak secure domination number and it is denoted respectively as $\gamma_{ws}(G)$. If there is no confusion we use γ for $\gamma_S(G)$ throughout this paper.

One may observe that $\gamma \leq \gamma_{st} \leq \gamma_{ss}$and $\gamma \leq \gamma_w \leq \gamma_{ws}$. The existence of a strong secure domination (wsd) is guaranteed since $V(G)$ is a SSDS (WSDS) of G.

The reader may recall that the corona product of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H and then joining the i^{th} vertex of G to every vertex in the i^{th}copy of H, where 1≤ i≤ n and it is denoted by $G \circ H$. Following known results are used in this paper.

Theorem 1.1 [1] For the path P_n, $\gamma_s(P_n)$= $\left\lceil\frac{3n}{7}\right\rceil$

Theorem 1.2 [1] For the cycle C_n, $\gamma_s(C_n)$= $\left\lceil\frac{3n}{7}\right\rceil$

Theorem 1.3 [1] For the complete graph K_n, $\gamma_s(K_n)$=1

Theorem 1.4 [1] A set X is a secure dominating set if and only if for each u in $V - X$, there exists v in X such that $G[P_n(v, X) \cup \{u, v\}]$ is complete.

Theorem 1.5 [6] For any tree T with k −support vertices and e pendant vertices $\gamma_{st} \geq k$ and $\gamma_w \geq e$.

2. Strong and Weak secure domination number of some classes of graphs

One may observe that the two parameters strong domination number (γ_{st}) and secure domination number (γ_s) are not comparable.

For example, consider $K_{1,3}$ with its vertex set as $V = \{x, v_1, v_2, v_3\}$ where x is the centre vertex. Then $\{x\}$ is a strong dominating set but not a secure dominating set. And $S = \{v_1, v_2, v_3\}$ is a secure dominating set but not a strong dominating set.

Also, weak domination number (γ_w) and secure domination number (γ_s) are not comparable. For the wheel graph W_6, with $V(W_6) = \{v, v_1, v_2, v_3, v_4, v_5, v_6\}$, the set $S = \{v_2, v_3, v_4\}$ is a weak dominating set but not a secure dominating set.

Now in Figure 1, the set $S = \{2,3,6\}$ is a secure dominating set but not a weak dominating set.

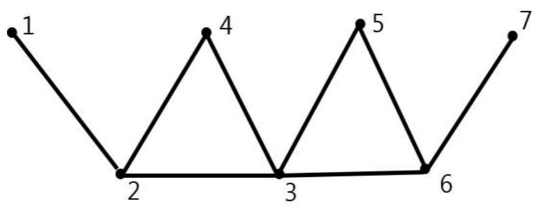

Fig. 1

It is immediate from the definition that $\gamma_{ss}(K_n) = \gamma_{ws}(K_n) = 1$.

Theorem 2.1 For any path P_n, $\gamma_{ss}(P_n) = \gamma_{ws}(P_n) = \left\lceil\frac{3n}{7}\right\rceil$

Proof Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Invoking Theorem 1.1, $\gamma_{ss}(P_n) \geq \gamma_s(P_n) = \left\lceil\frac{3n}{7}\right\rceil$. Hence it is sufficient to prove that $\gamma_{ss}(P_n) \leq \left\lceil\frac{3n}{7}\right\rceil$. Let $n = 7r + s; r \geq 1$ and $0 \leq s \leq 6$.

Consider $R = \bigcup_{i=0}^{k-1}\{v_{7i+2}, v_{7i+4}, v_{7i+6}\}$ and

$$S = \begin{cases} \emptyset \ if\ s = 0 \\ \{v_{7k+1}\}\ if\ s = 1,2 \\ \{v_{7k+1}, v_{7k+3}\}\ if\ s = 3,4 \\ \{v_{7k+1}, v_{7k+3}, v_{7k+5}\}\ if\ s = 5,6. \end{cases}$$

Then by Theorem 1.1, $R \cup S$ is a secure dominating set of P_n. $R \cup S$ is also a strong secure dominating set of P_n. For, each v_i in $V - \{R \cup S\}$, there exists some $v_j (i \neq j)$ such that $d(v_i) \leq d(v_j)$, which implies $\gamma_{ss}(P_n) \leq \left\lceil \frac{3n}{7} \right\rceil$.

We have by Theorem 1.1, $\gamma_{ws}(P_n) \geq \gamma_s(P_n) = \left\lceil \frac{3n}{7} \right\rceil$. Now to prove that $\gamma_{ws}(P_n) \leq \left\lceil \frac{3n}{7} \right\rceil$, we consider two cases. (i) If $R \cup S$ contains no support vertex, then its corresponding pendant vertex belongs to S.

(ii) If $R \cup S$ contains no pendant vertex, then the support vertices u_2 and u_{n-1} belong to $R \cup S$. In this case, $((R \cup S) - \{u_2\}) \cup \{u_1\}$ or $((R \cup S) - \{u_{n-1}\}) \cup \{u_n\}$ is a WSDS. Then $\gamma_{ws}(P_n) \geq \left\lceil \frac{3n}{7} \right\rceil$, which in turn implies, $\gamma_{ws}(P_n) = \left\lceil \frac{3n}{7} \right\rceil$.

Theorem 2.2 For any cycle C_n, $\gamma_{ss}(C_n) = \gamma_{ws}(C_n) = \left\lceil \frac{3n}{7} \right\rceil$

Proof Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Invoking Theorem 1.2, $\gamma_{ss}(C_n) = \gamma_{ws}(C_n) \geq \left\lceil \frac{3n}{7} \right\rceil$. Since each vertex $v_i \in V - X$ is of degree 2, each secure dominating set is a SSDS as well a WSDS. Therefore, $\gamma_{ss}(C_n) = \gamma_{ws}(C_n) \leq \left\lceil \frac{3n}{7} \right\rceil$.

Theorem 2.3 $\gamma_{ss}(W_{n+1}) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil\ if\ n \equiv 1,2\ (mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1\ if\ n \equiv 0 (mod\ 3) \end{cases}$

Proof. Let $V(W_{n+1}) = \{x, v_1, v_2, \dots, v_n\}$, where x is the center vertex of the wheel and v_i' s are on the boundary of the wheel. For $n = 3, \gamma_{ss} = 1$. Let $n \geq 4$. Every SSDS contains the center vertex x and any dominating set S that contains x is a strong dominating set.

Consider the set $S = \begin{cases} \{x, v_2, v_5, v_8, \dots, v_{n-2}\}\ if\ n \equiv 1,2\ (mod\ 3) \\ \{x, v_2, v_5, v_8, \dots, v_{n-4}, v_{n-1}\}\ if\ n \equiv 0 (mod\ 3) \end{cases}$

Then S is a minimum SSDS of W_{n+1}, since each vertex v_i in S can defend at most two vertices namely v_{i-1} and v_{i+1} in $V - S$ and the vertex x can be the only defender for atmost two neighbouring vertices in $V - S$. Also $|S| = \left\lceil \frac{n}{3} \right\rceil\ if\ n \equiv 1,2\ (mod\ 3)$ and $\left\lceil \frac{n}{3} \right\rceil + 1\ if\ n \equiv 0 (mod\ 3)$.

Lemma 2.4 Let $n = 3k + 1, l \geq 9$ and l be an odd integer. Then $(k + 1) - \left\lceil \frac{l}{2} \right\rceil < \left\lceil \frac{3k+1-l}{3} \right\rceil$

Theorem 2.5 $\gamma_{ws}(W_{n+1}) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil\ if\ n = 3,4,7 \\ \left\lceil \frac{n}{3} \right\rceil + 1\ otherwise \end{cases}$

Proof. Let $V(W_{n+1}) = \{x, v_1, v_2, \dots, v_n\}$, where x is the center of the wheel and $\{v_1, v_2, \dots, v_n\}$ form a cycle C_n in W_{n+1}. Then $\gamma_{ws}(W_4) = \gamma_{ws}(K_4) = 1$. Let X be a γ-set of C_n. Then $X \cup \{x\}$ is a weak secure dominating set of W_{n+1}, which implies that $\gamma_{ws}(G) \leq |X| + 1 = \left\lceil \frac{n}{3} \right\rceil + 1$. Let $n \geq 4$ and S' be a γ_{ws} set of W_{n+1}. If $x \notin S'$, then $S' \subseteq \{v_1, v_2, \dots, v_n\}$. So S' becomes a dominating set of C_n. Thus $\left\lceil \frac{n}{3} \right\rceil = \gamma(C_n) \leq |S'|$. If $x \in S'$, then $V - S' \subseteq \{v_1, v_2, \dots, v_n\}$. Since $\deg x > \deg v_i$ for every i, $(n \geq 4)$ no vertex in $V - S'$ is dominated by x weakly. Hence it is necessarily to have one vertex in C_n to dominate every vertex in $V - S'$ weakly, implying $S' - \{x\}$ is a dominating set of C_n. Hence $\left\lceil \frac{n}{3} \right\rceil = \gamma(C_n) \leq |S' - \{x\}|$.

$$\text{Therefore, } |S'| \geq \left\lceil \frac{n}{3} \right\rceil + 1 > \left\lceil \frac{n}{3} \right\rceil \qquad (1)$$

Hence $\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{ws} \leq \left\lceil \frac{n}{3} \right\rceil + 1$. For $n = 4, S = \{v_1, v_3\}$; $n = 7, S = \{v_2, v_5, v_7\}$ is a WSDS with cardinality $\left\lceil \frac{n}{3} \right\rceil$. For $n = 6$ or $n \geq 8$, consider a γ set of C_n. By equation (1), $x \in V - S$ and $S \subseteq \{v_1, v_2, \dots, v_n\}$. For $n = 3k, |S| = k$ and these k vertices have to dominate $3k$ vertices in C_n. So every vertex in S has two nonadjacent private neighbours in $V - S$. In view of Theorem 1.4, S is not a secure dominating set. Therefore $|S|=\left\lceil \frac{n}{3} \right\rceil + 1$.

When $n = 3k + 1$, S is a γ set of C_n with $k + 1$ elements. Then any one of the following conditions must be true.

(i)There exist two vertices in S that are adjacent in C_n.

(ii)There exist two nonadjacent vertices in S that are adjacent to a vertex in C_n.

Suppose (i) true. Then the adjacent vertices in S can dominate exactly four vertices in C_n. The remaining $3k - 3$ vertices in C_n are to be dominated by $k - 1$ vertices in S. So there exists a vertex that has two nonadjacent private neighbours in S which contradicts Theorem 1.4. Suppose (ii) is true. Let the longest path in $C_n$whose alternate vertices are in S be denoted by P_l. Then the end vertices of P_l are in $V - S$ and l is odd, $l \leq 7$. For $l \geq 9$, there are $\left\lceil \frac{l}{2} \right\rceil$ vertices of P_l are in S. So there are only $(k + 1) - \left\lceil \frac{l}{2} \right\rceil$ vertices of S to dominate the remaining vertices of C_n, a path P_{n-l}. However we need atleast $\left\lceil \frac{3k+1-l}{3} \right\rceil$ vertices to dominate the remaining vertices of P_{n-l}. Then $(k + 1) - \left\lceil \frac{l}{2} \right\rceil < \left\lceil \frac{3k+1-l}{3} \right\rceil$ when $l \geq 9$. So this is not possible. Therefore, we conclude that the only three possibilities are $l = 3,5$ and 7. For $l = 3$, there exists one vertex that has two nonadjacent private neighbours which contradicts Theorem 1.4. For $l = 5, S = \{v_2, v_4\}$ *is* not a secure dominating set (Refer Figure 2).

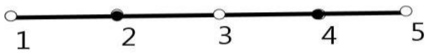

Fig. 2

For, v_4 is the private neighbour of v_5 and v_2 is the private neighbour of v_1 and so $v_3$has no defender. For $l = 7, S = \{v_2, v_4, v_6\}$. (Refer Figure 3).

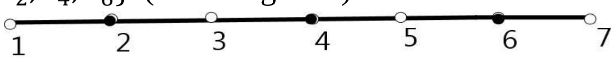

Fig. 3

The remaining $3(k - 2)$ vertices in C_n are to be dominated by $k - 2$ vertices. In this case there exists atleast one vertex that has two non adjacent private neighbours, a contradiction to Theorem 1.4. Therefore $|S| = \left\lceil \frac{n}{3} \right\rceil + 1$. A similar argument follows for $n = 3k + 2$. Therefore in all the cases $\gamma_{ws}(W_{n+1}) = \left\lceil \frac{n}{3} \right\rceil + 1$. However for $n = 5$ and any γ −set S, there exists a vertex in $V - S$ that has no defender. Therefore $|S| = \left\lceil \frac{n}{3} \right\rceil + 1$.

Theorem 2.6 For the complete bipartite graph $K_{p,q}$, where $p \leq q$

$$\gamma_{ss} = \begin{cases} q \; if \; p = 1 \\ p \; if \; p \geq 2 \; and \; p < q \\ p \; if \; p = 2,3 \; and \; p = q \\ 4 \; if \; p \geq 4 \; and \; p = q \end{cases}$$

$$\gamma_{ws} = \begin{cases} q \; if \; p \geq 1 \; and \; p < q \\ p \; if \; p = 2,3 \; and \; p = q \\ 4 \; if \; p \geq 4 \; and \; p = q \end{cases}$$

Proof. Let $P = \{v_1, v_2, \dots, v_p\}$ and $Q = \{u_1, u_2, \dots, u_q\}$ be the bipartite sets of $K_{p,q}$.

Case (i) p=1

$\gamma_{ws} = q$. Since every SSDS of $K_{1,q}$ must contain its support vertex and its $q-1$ pendant vertices, $\gamma_{ss} = q$.

Case (ii) $p \geq 2$ and $p < q$

The set P is a minimum ssds and Q is a minimum wsds with $\gamma_{ss} = p$ and $\gamma_{ws} = q$.

Case (iii) $p = 2,3$ and $p = q$

If $p = 2$, then $X = \{v_1, u_1\}$ is a SSDS and WSDS. If $p = 3, X = \{u_1, v_1, v_2\}$ is a ssds and wsds.

Case (iv) $p \geq 4$ and $p = q$

Let $X = \{v_1, v_2, u_1, u_2\}$ is a secure dominating set. Since $p = q, X$ is a SSDS and WSDS. Therefore $\gamma_{ss} = \gamma_{ws} = 4$.

3. Strong Weak Secure domination number and Related results

Following two lemma's are immediate.

Lemma 3.1 For any graph $G, \gamma_{ss}(G) = 1$ if and only if $G \cong K_n$

Lemma 3.2 For any tree $T, \gamma_{ss}(T) = 2$ if and only if $T \cong P_3$ or $T \cong P_4$

Theorem 3.3 Let G be a connected graph.

(i) If there is a vertex u in G such that $d(v) < d(u)$ for every $v \epsilon N(u)$, then u belongs to every SSDS of G.

(ii) If G has a unique vertex of full degree, then it belongs to every SSDS of G.

(iii) If $G$is not complete and has at least two vertices of full degree, then $\gamma_{ss} = 2$.

Proof. (i) Let G be a graph with a vertex u such that $d(v) < d(u)$ for every $v \epsilon N(u)$. Then, u belongs to every SSDS. For otherwise, there is no vertex in G that strongly dominates u.

(ii) Proof follows from (i).

(iii) When G is not complete, $\gamma_{ss} \geq 2$. Let u_1 and u_2 be the two full degree vertices. Then $X = \{u_1, u_2\}$ is a SSDS of G.

Theorem 3.4 Let G be any graph with n vertices and $S = \{ v_1, v_2, \dots, v_k\}$ be the set of $k-$support vertices of G. Then $\gamma_{ss} \leq n - k$.

Proof. Let $k \geq 1$. Choose a pendant vertex u_i, $1 \leq i \leq k$ corresponding to each support vertex v_i, $1 \leq i \leq k$. Take $X = \{u_1, u_2, \dots, u_k\}$. Then $V - X$ is a SSDS. Each u_i is strongly defended by its corresponding support vertex. Hence $\gamma_{ss} \leq n - k$.

Now, we characterize the classes of graphs with k support vertices having $\gamma_{ss}(G) = n - k$.

Theorem 3.5 Let G be any graph with n vertices and $S = \{v_1, v_2, \dots, v_k\}$ be the set of k-support vertices of G. Then $\gamma_{ss}(G) = n - k$ iff $N(v) \subseteq S$ for every $v \in V - S$.

Proof. Assume that $\gamma_{ss} = n - k$.Without loss of generality let $k \geq 1$. Let $S = \{v_1, v_2, \dots, v_k\}$ be the set of support vertices. Choose a pendant vertex u_i corresponding to each v_i and let $X =$

$\{u_1, u_2, \ldots, u_k\}$. Then $V - X$ is a γ_{ss}-set of G. Let $v \in V - S$. We prove that $N(v) \subseteq S$ for every $v \in V - S$. Suppose not. Then there exists $u \in V - S$ such that u and v are adjacent and both are not pendant vertices. Without loss of generality let $d(u) \leq d(v)$. Then $V - (X \cup \{u\})$ is a strong dominating set. In this set the defender for every u_i is v_i and u is v. Therefore $V - (X \cup \{u\})$ is a SSDS of G and $\gamma_{ss} \leq n - (k+1)$, which is a contradiction. Hence $N(v) \subseteq S$, for every $v \in V - S$. On the other hand $N(v) \subseteq S$, for every $v \in V - S$. Suppose $\gamma_{ss} < n - k$. Then $|V - X'| \geq k + 1$, where X' is a minimum SSDS of G. Since there are atleast $k + 1$ vertices in $V - X'$, there must exist atleast one vertex that is not a support vertex in $V - X'$. In $V - X'$, if possible let there exists l vertices that are not support vertices, where $l \geq 1$ and the remaining are support vertices. Clearly $V - X'$ has at least $k + l - 1$ support vertices. In view of Theorem 1.4, no two of the l vertices have the same defender. So, $V - X'$ can have atmost $k - l$ support vertices, a contradiction to the fact that $V - X'$ has atleast $k + 1 - l$ support vertices. Hence $\gamma_{ss}(G) = n - k$.

Corollary 1. $\gamma_{ss}(K_{1,k}) = k$

Corollary 2. $\gamma_{ss}(P_4) = 2$ and $\gamma_{ss}(P_5) = 3$

Corollary 3. $\gamma_{ss}(G \circ K_1) = n$ if G is a graph with n vertices.

Theorem 3.6. Let G be any graph with n vertices and $S = \{v_1, v_2, \ldots, v_k\}$ be the set of k −support vertices of G. Then $\gamma_{ws} \leq n - k$.

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the set of $k -$ support vertices of G. Then $V - S$ is a weak dominating set. The defender for each support vertex is one of its pendant neighbours. Therefore $\gamma_{ws} \leq n - k$.

Theorem 3.7. (Nordhaus-Gaddum type results) Let T be a tree with $n \geq 3$ vertices, k supports and e pendant vertices. Then

$$k \leq \gamma_{ss}(T) \leq n - k \quad (2)$$
$$e \leq \gamma_{ws}(T) \leq n - k \quad (3)$$

Further if $T \neq K_{1,n}$, then

$$2 \leq \overline{\gamma_{ss}}(T) \leq n - k \quad (4)$$
$$\overline{\gamma_{ws}}(T) \leq n - e \quad (5)$$
$$k + 2 \leq \gamma_{ss}(T) + \overline{\gamma_{ss}}(T) \leq 2(n - k) \quad (6)$$
$$\gamma_{ws}(T) + \overline{\gamma_{ws}}(T) \leq 2n - (k + e) \quad (7)$$

In all the above inequalities, the equality is attained for the graph P_4.

Proof. By Theorem 1.5 and by Theorem 3.5, (2) holds. By Theorem 1.5 and by Theorem 3.6 (2) holds. To establish (4), let $T \neq K_{1,n}$. Then T has atleast two pendant vertices and two supports. Each pendant vertex in T is adjacent to $n - 2$ vertices in $\bar{T}$. Since $\bar{T}$ has a vertex of full degree, $\bar{\gamma}(T) \geq 2$. Thus $\overline{\gamma_{ss}}(T) \geq \overline{\gamma_s}(T) \geq 2$. Lower bound (4) holds. Let $X = \{v_1, v_2, \ldots, v_k\}$ be the set of support vertices of T. Then $V - X$ is a strong dominating set for $\bar{T}$. Since each tree has at least two pendant vertices, the defender for every v_i in $\bar{T}$ is one of the pendant neighbours of v_j in T, where $i \neq$j. Hence $V - X$ is a SSDS of $\bar{T}$. Upper bound (4) holds. Let $E = \{u_1, u_2, \ldots, u_e\}$ be the set of pendant vertices of $\bar{T}$. Each pendant vertex in T has maximum degree $n - 2$ in $\bar{T}$. Then $V - E$ is a weak dominating set. In $\bar{T}$, all the pendant vertices of T form a clique of size e, the defender for every u_i in $\bar{T}$ is one of the support vertices v_j in T, where $i \neq j$. Then $V - E$ is a WSDS of $\bar{T}$. So (5) holds. (6) can be derived from (2) and (4). (3) and (5) gives (7).

Consider P_4. Since P_4 is self complementary, equality is attained.

Theorem 3.8. For any graph G of order $n, \gamma_s \leq \gamma_{ss} \leq n-1$ and $\gamma_w \leq \gamma_{ws} \leq n-1$. The bound is attained for star graph.

Proof. Let v be a vertex of maximum degree Δ. Choose a vertex u that is adjacent to v. Then $V-\{u\}$ is a SSDS and $V-\{v\}$ is a WSDS. So $\gamma_{ss} \leq n-1$ and $\gamma_{ws} \leq n-1$. For the star graph $K_{1,n}$, $\gamma_{ws}(G) = \gamma_{ss}(G) = n-1$.

4. Conclusion

In this paper, we initiate a study on SSDS and WSDS. There is a wide scope for further investigation on these parameters for many other graph classes, graph operations and graph products. The strong (weak) secure domination problem is to determine a minimum strong (weak) secure dominating set of G. Computing the complexity of decision version of the strong (weak) secure domination problem is an important problem while considering the various applications of these parameters in security analysis and Benes network. Investigations of such problems are to be considered in a separate paper.

References

[1] EJ Cockayne, PJP Grobler, WR grundlingh, J Munganga and JH van Vuuren, "Protection of a Graph", Discrete Mathematics, Util Math 67, 2005, pp. 19-32..

[2] Claude Berge, "The Theory of Graphs", Dover Publications, INC, Mineola, New York, 1962.

[3] F. Harary, "Graph Theory", Narosa Publ. House, New Delhi, 2001.

[4] Teresa W. Haynes, Stephen T. Hedetnimi, Peter J. Slater, "Fundamentals of domination in graphs", Marcel Dekkar, Inc, New York, Basel, 2013.

[5] S.V.D Rashmi, S. Arumugam, K.R. Bhutani and P. Gartland, "Perfect secure domination in graphs", Categories and General Algebraic Structures with applications, July 2017, pp. 125-140.

[6] E.S. Kumar, L.P. Latha, "Strong weak domination and domination balance in a graph", Discrete Mathematics 161, 1996, pp.235-242.

[7] Nouby M. Ghazaly, M. M. A. . (2022). A Review on Engine Fault Diagnosis through Vibration Analysis . International Journal on Recent Technologies in Mechanical and Electrical Engineering, 9(2), 01–06. https://doi.org/10.17762/ijrmee.v9i2.364

[8] M.R. Farahani, "On Domination Polynomial of TUC4C8(S) Nanotube". Pacific Journal of Applied Mathematics. 7(2), 2015, pp.1-5.

[9] Bulla, P. . "Traffic Sign Detection and Recognition Based on Convolutional Neural Network". International Journal on Recent and Innovation Trends in Computing and Communication, vol. 10, no. 4, Apr. 2022, pp. 43-53, doi:10.17762/ijritcc.v10i4.5533.

[10] M.R. Farahani, H. Hosseini. "A new method for computing Domination Polynomials of graphs Capra Ca(C_6)". Pacific Journal of Applied Mathematics. 7(3), 2015, pp. 6-10

[11] M.R. Farahani, "Domination polynomial of Nanotorus by Using the 2-variables Generating Function". Pacific Journal of Applied Mathematics. 6(1), 2014, pp.79-95.

[12] S. Hussain, F. Afzal, D. Afzal, M.R. Farahani, M. Cancan, S. Ediz, "Theoretical study of benzene ring embedded in P-type surface in 2D network using some new degree based topological indices via M-polynomial". Eurasian Chem. Commun. 3 (2021) pp. 180-186. https://doi.org/10.22034/ECC.2021.272315.1134