Power Series Methods for Solving Fractal Differential Equations using Least Squares Approximations

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Issue: *Special Issue on Mathematical Computation in Combinatorics and Graph Theory in Mathematical Statistician and Engineering Applications*

Article Info
Page Number: 115 - 129
Publication Issue:
Vol 71 No. 3s3 (2022)

Article History
Article Received: *30 April 2022*
Revised: *22 May 2022*
Accepted: *25 June 2022*
Publication: *02 August 2022*

Abstract
For introducing of studying the fractal differential Equations, a proposed method for its solution was studied using Power series Methods via least squares approximations. The method depend on the definitions of Riemann-liouville fractional derivatives. The linear non homogenous fractal differential Equations are solved with detailed examples. For checking the errors, many comparisons of solutions with exact solutions and with known solutions.

Keywords: Riemann-liouville, fractal differential Equations, Power series methods, least squares approximations, Fractional residual power series.

2010 AMS: 40Cxx, 40C19

1. Introduction

Fractional calculus theory is a mathematical analysis tool applied to the study of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration [1,2,3,4]. Commonly these fractional integrals and derivatives were not known to many scientists and up until recent years, they have been only used in a purely mathematical context, but during these last few decades these integrals and derivatives have been applied in many science contexts due to their frequent appearance in various applications in the fields of fluid mechanics, viscoelasticity, biology, physics, image processing, entropy theory, and engineering [5,6,7,8] It is well known that the fractional order differential and integral operators are non-local operators. This is one reason why fractional calculus theory provides an excellent instrument for description of memory and hereditary properties of various physical processes. For example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [1,2,3,4]. Applying fractional calculus theory to entropy theory has also become a significant tool and a hotspot research domain [9,10] Power series have become a fundamental tool in the study of elementary functions and also other not so elementary ones as can be checked in any book of analysis. They have been widely used in computational science for easily obtaining an approximation of functions [11] In physics, chemistry, and many other sciences this power expansion has allowed scientist to make an approximate study of many systems, neglecting higher order terms around the equilibrium point.

This is a fundamental tool to linearize a problem, which guarantees easy analysis[12,13] The study of fractional derivatives presents great difficulty due to their complex integro-differential definition, which makes a simple manipulation with standard integer operators a complex operation that should be done carefully. The solution of fractional differential equations (FDEs), in most methods, appears as a series solution of fractional power series (FPS) [14,15,16,17] We will study a developed solution method, which is the combination of two methods of the power series and the approximation of least squares with giving some examples and solving them in the developed way and comparing with the real solution as shown in the drawing and the attached tables with examples and programmed in Matlab.

2-Basic concepts of fractal differential equations

Defintion (1):In the theory of fractional calculus, the entire gamma function Γ(t) plays a substantial turn. An inclusive definition of Γ(t) is that supplied by Euler limit [17]

$\Gamma(t) = \lim_{N\to\infty} \frac{N!Nt}{t(t+1)(t+2)(t+N)}, t > 0$ (1)

Whilst the beneficial integral convert form is defined by:

$\Gamma(t)=\int_0^\infty y^{t-1} e^{-u}\, du, R(t) > 0$ (2)

Defintion(2):The α th order left and right Riemann-liouville integrals of funaction y(x) are defined on the interval (a,b) as following[1][18][22]:

$aI_x^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{y(s)}{(x-s)^{1-\alpha}}\, ds$ (3)

$xI_b^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{y(s)}{(x-s)^{1-\alpha}}\, ds$ (4)

Where $\alpha > 0$ are called fractional integrals of the order α, they are sometimes called left-sides and right-sided fractional integrals respectively.

Defintion (3) : The α th order left and right Riemann-liouville drrivative of funaction y(t) are defined on the interval (a,b) are given as[1, 18, 22]

RL $D_{a,x}^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\tau)^{n-\alpha-1}\, y(\tau) d\tau$ (5)

RL $D_{x,b}^\alpha y(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (\tau-x)^{n-\alpha-1}\, y(\tau) d\tau$ (6)

where $n-1 < \alpha < n \in z^+$.

2.1:the composition of Riemann-liouville fractional integral and derivative

- If $F \in C[0,\infty)$,then the Riemann-liouville fractional order integral has the following important property [1],[18]:

$I^\alpha(I^\beta f(x)) = I^\beta(I^\alpha f(x)) = I^{\alpha+\beta} f(x)$ where $\alpha > 0$ *and* $\beta > 0$

$$= \frac{1}{\Gamma(\alpha+\beta)} \int_0^x (x-t)^{\alpha+\beta-1}\, f(t)\, dt$$

- let use consider the fractional derivative of order α a fractional derivative of order β [2],[19]

$$D^\alpha(D^\beta f(t)) = D^{\alpha+\beta} f(t)$$

- for $\alpha > 0, t > 0$ [2],[3]

$D^\alpha(I^\alpha f(t)) = f(t)$

- $D^{\alpha}\big(\lambda f(t)+\mu g(t)\big)=\lambda D^{\alpha}f(t)+\mu D^{\alpha}g(t)$ [2]
- $D^{\alpha}\big(kf(t)\big)=kD^{\alpha}f(t)\ \alpha>0$
- $D^{\alpha}\big(f(t).g(t)\big)=[D^{\alpha}f(t)].g(t)+f(t)[D^{\alpha}\mathrm{g(t)}]$[2]
- $D^{\alpha}\big(f(t)+g(t)\big)=D^{\alpha}f(t)+D^{\alpha}g(t)\ \alpha\in R$

2.2 The derivative of fractal differential equation[2][18]

- $\mathrm{D}^{\alpha}(\mathrm{x}^{\mathrm{n}})=\frac{\Gamma(\mathrm{n}-1)}{\Gamma(\mathrm{n}-\alpha+1)}\ \mathrm{x}^{\mathrm{n}-\alpha}$
- $\mathrm{D}^{\alpha}(\sin\alpha\mathrm{x})=\mathrm{a}^{\alpha}\sin(\mathrm{ax}+\frac{\pi}{2}\alpha)$
- $\mathrm{D}^{\alpha}(\cos\alpha\mathrm{x})=\mathrm{a}^{\alpha}\cos(\mathrm{ax}+\frac{\pi}{2}\alpha)$
- $\mathrm{D}^{\alpha}\left(\mathrm{e}^{\mathrm{kx}}\right)=\mathrm{k}^{\alpha}\ \mathrm{e}^{\mathrm{kx}}$
- $\mathrm{D}^{\alpha}(\mathrm{C})=\frac{\mathrm{Cx}^{-\alpha}}{\Gamma(1-\alpha)}$

2.3:leaste square approximation:

Definition(4) Another approach to approximating a function f(x) on an interval a ≤ x ≤ b is to seek an approximation p(x) with a small 'average error' over the interval of approximation. A convenient definition of the average error of the approximation is given by[17][18]

$$E(p;\ f)\equiv\left[\frac{1}{b-a}\int_{a}^{b}[f(x)-p(x)]\right] \quad (7)$$

This is also called the root-mean-square-error (denoted subsequently by RMSE) in the approximation of f(x) by p(x). Note first that choosing p(x) to minimize E(p; f) is equivalent to minimizing

$$\int_{a}^{b}[f(x)-p(x)]^{2}\,dx \quad (8)$$

thus dispensing with the square root and multiplying fraction (although the minimums are generally different). The minimizing of (8) is called the least squares approximation problem For a given function

2.4:solution of homogeneous fractal differential eqution

The Power series is a fundamental tool in the study of elementary functions. They have been widely used in computational science for easily obtaining an approximation of functions. In thermal physics and many other sciences this power expansion has allowed scientist to make an approximate study of many differential equations

Theorem(1): Suppose that u(t) has a FPS representation at t = t0 of the form [19]:

$$y(t)=\sum_{\mathrm{n}=0}^{\infty}\mathrm{c_n}\,(\mathrm{t}-\mathrm{t}_\circ)^{\mathrm{n}\alpha}=\mathrm{c}_\circ+\mathrm{c}_1(\mathrm{t}-\mathrm{t}_\circ)^{\alpha}+\mathrm{c}_2(t-t_\circ)^{2\alpha}+\cdots(n\ time\ ) \quad (9)$$

Where $0\leq m-1<\alpha\leq m, m\in N^{+} and\ t\geq t_\circ$ is called afractional power series about $t_\circ$,where t is variable and $c_n$are the coefficients of series and will be of the form[24]

$$c_n=\frac{D^{n\alpha}y(t_\circ)}{\Gamma(n\alpha+1)} \quad (10)$$

Theorem(2): Suppose that the fractional power series $\sum_{n=0}^{\infty}c_n\,t^{n\alpha}$ has radius of convergence $R>0$. If $f(t)$ is afunaction defined by $f(t)=\sum_{n=0}^{\infty}c_n t^{n\alpha}$ on $0\leq t\leq R$,Then for $m-1<\alpha\leq m$ and $0\leq t\leq R$ we have[21][24]

$D^{\alpha}f(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-\alpha+1)}\ t^{(n-1)\alpha}$ (11)

$I^{\alpha}f(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+\alpha+1)}\ t^{(n+1)\alpha}$(12)

(2.4.1)Fractional residual power series method

we intend to use the FRPS method to solve a class of solid systems In fractional order by replacing FPS expansions within the remaining truncation functions. To do this, we assume that the FPS solution at t = 0 has the following form: The objective of the FRPS algorithm is to obtain an approximate solution supporting the the proposed model. Thus, using the initial conditions, it can be written as follows[25]

$y = \sum_{n=0}^{\infty} \frac{c^n t^{\alpha n}}{\Gamma(n\alpha+1)}$ (13)

And the solution approximation is

$y(t) = c_\circ + \sum_{n=1}^{\infty} \frac{c^n t^{\alpha n}}{\Gamma(n\alpha+1)}$. (14)

Example(1): consider the fractional differential equation

Solve $D^{\alpha}\ y - y = 0$

Solution*:* Suppose that $y = \sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}$ and We substitute it in the given equation[24].

$$let\ y = \sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}$$

$$D^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) - \left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} \frac{c^n}{\Gamma(n\alpha+1)} D^{\frac{1}{2}} x^{n\alpha}\right) - \left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} \frac{c^n}{\Gamma(n\alpha+1)} \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-\alpha+1)}\ x^{n\alpha}\right) - \left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} \frac{c^n}{\Gamma(\alpha(n-1)+1)}\ x^{n\alpha}\right) - \left(\sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}\right) = 0$$

Let $k = n - 1\ k = n$

$$\left(\sum_{n=0}^{\infty} \frac{c^{k+1}}{\Gamma(k\alpha+1)}\ x^{k\alpha}\right) - \left(\sum_{n=0}^{\infty} \frac{c^k x^{k\alpha}}{\Gamma(k\alpha+1)}\right) = 0$$

$$\sum_{n=0}^{\infty} c^{k+1} - c^k \left(\frac{x^{\alpha k}}{\Gamma(\alpha k+1)}\right) = 0 and\ \frac{x^{\alpha k}}{\Gamma(\alpha k+1)} \neq 0$$

$c^{k+1} - c^k = 0 \Rightarrow c^k(c-1) = 0\ but\ c^k \neq 0 \Rightarrow \boldsymbol{c = 1}$

$y(x) = c^{\circ} + c^1 \frac{x^{\alpha}}{\Gamma(\alpha+1)} + c^2 \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + c^3 \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots$

The genral solution is $y = \sum_{n=0}^{\infty} \frac{c^n x^{\alpha n}}{\Gamma(n\alpha+1)}$.

3. Suggested method for solving nonhomogeneous fractal differential equations power series with the use of least squares approximation

If we had a non-homogenuse linear fractal differential equation in the form for example , $\mathrm{D}^{\alpha}\mathrm{y}+\mathrm{by}=\mathrm{f(x)}$, it is solved by a power series method, which is to substitute for each $y=\sum_{n=0}^{\infty}\frac{c_n x^{n\alpha}}{\Gamma(n\alpha+1)}$ and simplify the expression

$$D^{\alpha}y+by=f(x)$$

$$y=\sum_{n=0}^{\infty}\frac{c_n x^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$D^{\alpha}\sum_{n=0}^{\infty}\frac{c_n x^{n\alpha}}{\Gamma(n\alpha+1)}+b\sum_{n=0}^{\infty}\frac{c_n x^{n\alpha}}{\Gamma(n\alpha+1)}=f(x)$$

$$\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(n\alpha+1)}D^{\alpha}x^{n\alpha}+b\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(n\alpha+1)}\,x^{n\alpha}=f(x)$$

$$\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(n\alpha+1)}\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)}\,x^{n\alpha-\alpha}+b\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(n\alpha+1)}\,x^{n\alpha}=f(x)$$

$$\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(n\alpha+1-\alpha)}\,x^{\alpha(n-1)}+b\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(n\alpha+1)}\,x^{n\alpha}=f(x)$$

$$\sum_{n=0}^{\infty}c_n\left[\frac{x^{\alpha(n-1)}}{\Gamma(n\alpha+1-\alpha)}+b\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}\right]=f(x)$$

We assume the $\left[\frac{x^{\alpha(n-1)}}{\Gamma(n\alpha+1-\alpha)}+b\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}\right]=w_n$

$$\sum_{n=0}^{\infty}c_n\,w_n=f(x)$$

$$c_{\circ}w_{\circ}+c_1w_1+c_2w_2+\cdots+c_nw_n=f(x)$$

By using least squares approximation

$\Delta=\int_a^b(T(y)-y)^2$ (15)

$T(y)=\sum_{n=0}^{\infty}c_nw_n-f(x)$(16)

$y=\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(n\alpha+1)}x^{n\alpha}=\sum_{n=0}^{\infty}c_n\,v_n$ (17)

We substitute (16), (17) in (15)

$$\Delta=\int_a^b\left[\left(\sum_{n=0}^{\infty}c_nw_n-f\right)-\sum_{n=0}^{\infty}c_n\,v_n\right]^2$$

$$\Delta=\int_a^b\left[\left(\sum_{n=0}^{\infty}c_n(w_n-v_n)-f\right)\right]^2$$

$$\Delta=\int_a^b(c_{\circ}\,(w_{\circ}{}_{-}v_{\circ})+c_1(w_{1-}v_1)+c_2(w_{2-}v_2)+\cdots+c_n(w_{n-}v_n)-f)^2$$

$$\frac{\partial\Delta}{\partial c_\circ} = 2\int_a^b (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + c_2(w_2 - v_2) + \cdots + c_n(w_n - v_n) - f)(w_\circ - v_\circ) = 0$$

$$\frac{\partial\Delta}{\partial c_1} = 2\int_a^b (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + c_2(w_2 - v_2) + \cdots + c_n(w_n - v_n) - f)(w_1 - v_1) = 0$$

$$\frac{\partial\Delta}{\partial c_2} = 2\int_a^b (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + c_2(w_2 - v_2) + \cdots + c_n(w_n - v_n) - f)(w_2 - v_2) = 0$$

$$\frac{\partial\Delta}{\partial c_n} = 2\int_a^b (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + c_2(w_2 - v_2) + \cdots + c_n(w_n - v_n) - f)(w_n - v_n) = 0$$

$T = w - v$

$0 = 2\int_a^b (c_\circ T_\circ + c_1T_1 + c_2T_2 + \ldots + c_nw_n - f)T_\circ$

$0 = 2\int_a^b (c_\circ T_\circ + c_1T_1 + c_2T_2 + \ldots + c_nw_n - f)T_1$

$0 = 2\int_a^b (c_\circ T_\circ + c_1T_1 + c_2T_2 + \ldots + c_nw_n - f)T_2$

$0 = 2\int_a^b (c_\circ T_\circ + c_1T_1 + c_2T_2 + \ldots + c_nw_n - f)T_n$

$0 = \int_a^b (c_\circ T_\circ + c_1T_1 + c_2T_2 + \ldots + c_nT_n - f)T_\circ$

$0 = \int_a^b (c_\circ T_\circ + c_1T_1 + c_2T_2 + + c_nT_n - f)T_1$

$0 = \int_a^b (c_\circ T_\circ + c_1T_1 + c_2T_2 + \ldots + c_nT_n - f)T_2$

$0 = \int_a^b (c_\circ T_\circ + c_1T_1 + c_2T_2 + \ldots + c_nT_n - f)T_n$

$0 = \int_a^b (c_\circ T_\circ T_\circ + c_1T_1T_\circ + c_2T_2T_\circ + \ldots + c_nT_nT_\circ - fT_\circ)$

$0 = \int_a^b (c_\circ T_\circ T_1 + c_1T_1T_1 + c_2T_2T_1 + \ldots + c_nT_nT_1 - fT_1)$

$0 = \int_a^b (c_\circ T_\circ T_2 + c_1T_1T_2 + c_2T_2T_2 + \ldots + c_nT_nT_2 - fT_2)$

$0 = \int_a^b (c_\circ T_\circ T_n + c_1T_1T_n + c_2T_2T_n + + c_nT_nT_n - fT_n)$

Let $n = 2 \Rightarrow c_\circ, c_1, c_2$

$$\int_a^b c_\circ T_\circ T_\circ + c_1T_1T_\circ + c_2T_2T_\circ = \int_a^b fT_\circ$$

$$\int_a^b c_\circ T_\circ T_1 + c_1T_1T_1 + c_2T_2T_1 = \int_a^b fT_1$$

$$\int_a^b c_\circ T_\circ T_2 + c_1T_1T_2 + c_2T_2T_2 = \int_a^b fT_2$$

$$\begin{bmatrix} \int_a^b T_\circ^2 & \int_a^b T_1T_\circ & \int_a^b T_2T_\circ \\ \int_a^b T_\circ T_1 & \int_a^b T_1^2 & \int_a^b T_2T_1 \\ \int_a^b T_\circ T_2 & \int_a^b T_1T_2 & \int_a^b T_2^2 \end{bmatrix} \begin{bmatrix} C_\circ \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int_a^b fT_\circ \\ \int_a^b fT_1 \\ \int_a^b fT_2 \end{bmatrix}$$

$$let \begin{bmatrix} \int_a^b T_\circ^2 & \int_a^b T_1T_\circ & \int_a^b T_2T_\circ \\ \int_a^b T_\circ T_1 & \int_a^b T_1^2 & \int_a^b T_2T_1 \\ \int_a^b T_\circ T_2 & \int_a^b T_1T_2 & \int_a^b T_2^2 \end{bmatrix} = A, \begin{bmatrix} \int_a^b fT_\circ \\ \int_a^b fT_1 \\ \int_a^b fT_2 \end{bmatrix} = B \ and \ \begin{bmatrix} C_\circ \\ c_1 \\ c_2 \end{bmatrix} = c$$

And the above described is solved to extract the unknown C values according to the equation
$(A^T.A)^{-1}A^T.B = C$ (18)

4. implimentations and comparisons of proposed method

We will make a comparison between the solution in the assumed way and the exact solution for the two examples shown below and through Figures (1) and (2) and the data table (1), (2) we will find the error value as shown in the mentioned tables

Example(2) solve the fractal differential$D^{\frac{1}{2}}y = x^2, \alpha = \frac{1}{2}$ $and\ n = 4, y(0) = 0$

$(D^{\frac{1}{2}}y = x^2) * D^{\frac{1}{2}}$

$D^{\frac{1}{2}}D^{\frac{1}{2}}y = D^{\frac{1}{2}}x^2$

$Dy = \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{2}{\frac{3\sqrt{\pi}}{4}}x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}}$

$$\int dy = \int \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}}\ dx$$

$y = \frac{8}{3\sqrt{\pi}}x^{\frac{5}{2}}\frac{2}{5} + c, c = 0$

$y = \frac{16}{15\sqrt{\pi}}x^{\frac{5}{2}}$ is the exact solution and solving the fractal differential eqution by the proposed method equation$D^{\frac{1}{2}}y = x^2$

let $y = \sum_{n=0}^{\infty}\frac{c_nx^{n\alpha}}{\Gamma(n\alpha+1)}$

$$D^{\frac{1}{2}} \sum_{n=0}^{\infty} (\frac{c_n x^{n\alpha}}{\Gamma(n\alpha + 1)}) = x^2$$

$$\sum_{n=0}^{\infty} (\frac{c_n}{\Gamma(n\alpha + 1)} D^{\frac{1}{2}} x^{n\alpha}) = x^2$$

$$\sum_{n=0}^{\infty} (\frac{c_n}{\Gamma(n\alpha + 1)} \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha - \alpha + 1)}) x^{\alpha(n-1)} = x^2$$

$$\sum_{n=0}^{\infty} (\frac{c_n}{\Gamma(n\alpha + 1)} \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha - \alpha + 1)}) x^{\alpha(n-1)} = x^2$$

$$\sum_{n=0}^{\infty} (\frac{c_n}{\Gamma(n\alpha - \alpha + 1)}) x^{\alpha(n-1)} = x^2$$

$\sum_{n=0}^{\infty} c_n \frac{x^{\alpha(n-1)}}{\Gamma(n\alpha-\alpha+1)} = x^2$let $w_n = \frac{x^{\alpha(n-1)}}{\Gamma(n\alpha-\alpha+1)}$

$$\sum_{n=0}^{\infty} c_n w_n = x^2$$

$$\Delta = \int_0^1 (T(y) - y)^2$$

$$T(y) = \sum_{n=0}^{\infty} c_n w_n - x^2 letv = \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)}$$

$\Delta = \int_0^1 (\sum_{n=0}^{\infty} c_n w_n - x^2 - \sum_{n=0}^{\infty} c_n v_n)^2$ $y = \sum_{n=0}^{\infty} c_n v_n$

$$\Delta = \int_0^1 (\sum_{n=0}^{\infty} c_n w_n - \sum_{n=0}^{\infty} c_n v_n - x^2)^2$$

$\Delta = \int_0^1 (\sum_{n=0}^{\infty} c_n w_n - c_n v_n - x^2)^2$

$$\Delta = \int_0^1 (\sum_{n=0}^{3} c_n (w_n - v_n) - x^2)^2 \; let \; n = 3$$

$$\Delta = \int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)^2$$

$$\frac{\partial \Delta}{\partial c_\circ} = 2\int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_\circ - v_\circ)$$

$$\frac{\partial \Delta}{\partial c_1} = 2\int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_1 - v_1)$$

$$\frac{\partial \Delta}{\partial c_2} = 2\int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_2 - v_2)$$

$$\frac{\partial\Delta}{\partial c_3} = 2\int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_3 - v_3)$$

$$0 = 2\int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_\circ - v_\circ)$$

$$0 = 2\int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_1 - v_1)$$

$$0 = 2\int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_2 - v_2)$$

$$0 = 2\int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_3 - v_3)$$

$$0 = \int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_\circ - v_\circ)$$

$$0 = \int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_1 - v_1)$$

$$0 = \int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_2 - v_2)$$

$$0 = \int_0^1 (c_\circ(w_\circ - v_\circ) + c_1(w_1 - v_1) + +c_2(w_2 - v_2) + c_3(w_3 - v_3) - x^2)(w_3 - v_3)$$

$let\ T = w - v$

$$0 = \int_0^1 (c_\circ T_\circ + c_1T_1 + c_2T_2 + c_3T_3 - x^2)T_\circ$$

$$0 = \int_0^1 (c_\circ T_\circ + c_1T_1 + c_2T_2 + c_3T_3 - x^2)T_1$$

$$0 = \int_0^1 (c_\circ T_\circ + c_1T_1 + c_2T_2 + c_3T_3 - x^2)T_2$$

$$0 = \int_0^1 (c_\circ T_\circ + c_1T_1 + c_2T_2 + c_3T_3 - x^2)T_3$$

$$\int_0^1 c_\circ T_\circ T_\circ + \int_0^1 c_1T_1T_\circ + \int_0^1 c_2T_2T_\circ + \int_0^1 c_3T_3T_\circ - \int_0^1 x^2\, T_\circ$$

$$\int_0^1 c_\circ T_\circ T_1 + \int_0^1 c_1T_1T_1 + \int_0^1 c_2T_2T_1 + \int_0^1 c_3T_3T_1 - \int_0^1 x^2\, T_1$$

$$\int_0^1 c_\circ T_\circ T_2 + \int_0^1 c_1 T_1 T_2 + \int_0^1 c_2 T_2 T_2 + \int_0^1 c_3 T_3 T_2 - \int_0^1 x^2\, T_2$$

$$\int_0^1 c_\circ T_\circ T_3 + \int_0^1 c_1 T_1 T_3 + \int_0^1 c_2 T_2 T_3 + \int_0^1 c_3 T_3 T_3 - \int_0^1 x^2\, T_3$$

$$\int_0^1 c_\circ T_\circ T_\circ + \int_0^1 c_1 T_1 T_\circ + \int_0^1 c_2 T_2 T_\circ + \int_0^1 c_3 T_3 T_\circ = \int_0^1 x^2\, T_\circ$$

$$\int_0^1 c_\circ T_\circ T_1 + \int_0^1 c_1 T_1 T_1 + \int_0^1 c_2 T_2 T_1 + \int_0^1 c_3 T_3 T_1 = \int_0^1 x^2\, T_1$$

$$\int_0^1 c_\circ T_\circ T_2 + \int_0^1 c_1 T_1 T_2 + \int_0^1 c_2 T_2 T_2 + \int_0^1 c_3 T_3 T_2 = \int_0^1 x^2\, T_2$$

$$\int_0^1 c_\circ T_\circ T_3 + \int_0^1 c_1 T_1 T_3 + \int_0^1 c_2 T_2 T_3 + \int_0^1 c_3 T_3 T_3 = \int_0^1 x^2\, T_3$$

$$\begin{bmatrix} \int_0^1 T_\circ T_\circ & \int_0^1 T_1 T_\circ & \int_0^1 T_2 T_\circ & \int_0^1 T_3 T_\circ \\ \int_0^1 T_\circ T_1 & \int_0^1 T_1 T_1 & \int_0^1 T_2 T_1 & \int_0^1 T_3 T_1 \\ \int_0^1 T_\circ T_2 & \int_0^1 T_1 T_2 & \int_0^1 T_2 T_2 & \int_0^1 T_3 T_2 \\ \int_0^1 T_\circ T_3 & \int_0^1 T_1 T_3 & \int_0^1 T_2 T_3 & \int_0^1 T_3 T_3 \end{bmatrix} \begin{bmatrix} C_\circ \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \int_0^1 x^2\, T_\circ \\ \int_0^1 x^2\, T_1 \\ \int_0^1 x^2\, T_2 \\ \int_0^1 x^2\, T_3 \end{bmatrix}$$

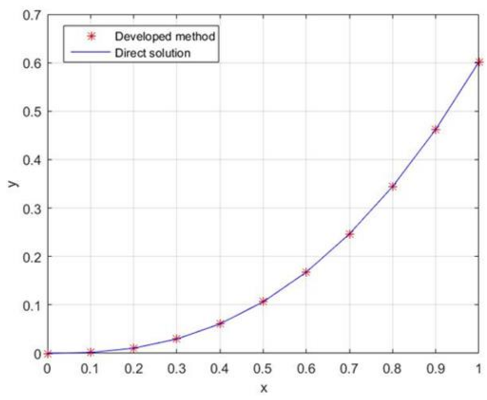

Figure (1): The approximate solution of Example (2) for some $\alpha = 0.5$

Example(3) solve the fractal differential$D^{\frac{1}{2}} y = x^2 - 2x, \alpha = \frac{1}{2}$ and $n = 3$, $y(0) = 0$

$D^{\frac{1}{2}} y = x^2 - 2x$

$y = \frac{16}{15\sqrt{\pi}} x^{\frac{5}{2}} + \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}}$ is the exact solution and solving the fractal differential eqution by the

Suggested method equation$D^{\frac{1}{2}} y = x^2 - 2x$ (1)

let $y = \sum_{n=0}^{\infty} \frac{c_n x^{n\alpha}}{\Gamma(n\alpha+1)}$ (2)

$$D^{\frac{1}{2}} \sum_{n=0}^{\infty} (\frac{c_n x^{n\alpha}}{\Gamma(n\alpha+1)}) = x^2 - 2x$$

$$\sum_{n=0}^{\infty} (\frac{c_n}{\Gamma(n\alpha+1)} D^{\frac{1}{2}} x^{n\alpha}) = x^2 - 2x$$

$$\sum_{n=0}^{\infty} (\frac{c_n}{\Gamma(n\alpha+1)} \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-\alpha+1)}) x^{\alpha(n-1)} = x^2 - 2x$$

Table (1) Approximate solutions for the example(2)

x	$y(Direct\ solution)$	$y(Developed\ method)$	$Error$
0.0000000	0.0000000	0.0000000	0.0000000
0.1000000	0.0019027	0.0019031	0.0000004
0.2000000	0.0107632	0.0107654	0.0000022
0.3000000	0.0296599	0.0296659	0.0000060
0.4000000	0.0608859	0.0608981	0.0000123
0.5000000	0.1063632	0.1063846	0.0000214
0.6000000	0.1677817	0.1678154	0.0000338
0.7000000	0.2466673	0.2467169	0.0000496
0.8000000	0.3444224	0.3444917	0.0000693
0.9000000	0.4623519	0.4624450	0.0000930
1.0000000	0.6016811	0.6018022	0.0001211

$$\sum_{n=0}^{\infty} (\frac{c_n}{\Gamma(n\alpha+1)} \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-\alpha+1)}) x^{\alpha(n-1)} = x^2 - 2x$$

$$\sum_{n=0}^{\infty} (\frac{c_n}{\Gamma(n\alpha-\alpha+1)}) x^{\alpha(n-1)} = x^2 - 2x$$

$\sum_{n=0}^{\infty} c_n \frac{x^{\alpha(n-1)}}{\Gamma(n\alpha-\alpha+1)} = x^2 - 2x$let $w_n = \frac{x^{\alpha(n-1)}}{\Gamma(n\alpha-\alpha+1)}$

$$\sum_{n=0}^{\infty} c_n w_n = x^2$$

$$\Delta = \int_0^1 (T(y) - y)^2 \ (1)$$

$$T(y)=\sum_{n=0}^{\infty}c_n w_n-(x^2-2x) let v=\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

$\Delta=\int_0^1(\sum_{n=0}^{\infty}c_n w_n-(x^2-2x)-\sum_{n=0}^{\infty}c_n\, v_n)^2\; y=\sum_{n=0}^{\infty}c_n\, v_n$

$$\Delta=\int_0^1(\sum_{n=0}^{\infty}c_n w_n-\sum_{n=0}^{\infty}c_n\, v_n-(x^2-2x))^2$$

$\Delta=\int_0^1(\sum_{n=0}^{\infty}c_n w_n-\ c_n v_n-(x^2-2x))^2$

$$\Delta=\int_0^1(\sum_{n=0}^{3}c_n(w_n-\ v_n)-(x^2-2x))^2\; let\; n=3$$

$$\Delta=\int_0^1(c_\circ(w_\circ-\ v_\circ)+c_1(w_1-v_1)+c_2(w_2-v_2)-(x^2-2x))^2$$

$$\frac{\partial\Delta}{\partial c_\circ}=2\int_0^1(c_\circ(w_\circ-v_\circ)+c_1(w_1-v_1)+c_2(w_2-v_2)-(x^2-2x))(w_\circ-v_\circ)$$

$$\frac{\partial\Delta}{\partial c_1}=2\int_0^1(c_\circ(w_\circ-v_\circ)+c_1(w_1-v_1)+c_2(w_2-v_2)-(x^2-2x)(w_1-v_1)$$

$$\frac{\partial\Delta}{\partial c_2}=2\int_0^1(c_\circ(w_\circ-v_\circ)+c_1(w_1-v_1)++c_2(w_2-v_2)\ -(x^2-2x)(w_2-v_2)$$

$$0=\int_0^1(c_\circ(w_\circ-v_\circ)+c_1(w_1-v_1)++c_2(w_2-v_2)-(x^2-2x)(w_\circ-v_\circ)$$

$0=\int_0^1(c_\circ(w_\circ-v_\circ)+c_1(w_1-v_1)++c_2(w_2-v_2)-(x^2-2x)(w_1-v_1)$

$0=\int_0^1(c_\circ(w_\circ-v_\circ)+c_1(w_1-v_1)++c_2(w_2-v_2)-(x^2-2x))(w_2-v_2)$

$let\; T=w-v$

$$0=\int_0^1(c_\circ T_\circ+c_1T_1+c_2T_2-(x^2-2x))T_\circ$$

$$0=\int_0^1(c_\circ T_\circ+c_1T_1+c_2T_2-(x^2-2x))T_1$$

$$0=\int_0^1(c_\circ T_\circ+c_1T_1+c_2T_2-(x^2-2x))T_2$$

$$\int_0^1 c_\circ T_\circ T_\circ+\int_0^1 c_1T_1T_\circ+\int_0^1 c_2T_2T_\circ=\int_0^1(x^2-2x)\,T_\circ$$

$$\int_0^1 c_\circ T_\circ T_1+\int_0^1 c_1T_1T_1+\int_0^1 c_2T_2T_1=\int_0^1(x^2-2x)\,T_1$$

$$\int_0^1 c_\circ T_\circ T_2 + \int_0^1 c_1 T_1 T_2 + \int_0^1 c_2 T_2 T_2 = \int_0^1 (x^2 - 2x)\, T_2$$

$$\begin{bmatrix} \int_0^1 T_\circ T_\circ & \int_0^1 T_1 T_\circ & \int_0^1 T_1 T_2 \\ \int_0^1 T_\circ T_1 & \int_0^1 T_1 T_1 & \int_0^1 T_1 T_2 \\ \int_0^1 T_\circ T_2 & \int_0^1 T_1 T_2 & \int_0^1 T_2 T_2 \end{bmatrix} \begin{bmatrix} C_\circ \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \int_0^1 (x^2 - 2x)\, T_\circ \\ \int_0^1 (x^2 - 2x)\, T_1 \\ \int_0^1 (x^2 - 2x)\, T_2 \end{bmatrix}$$

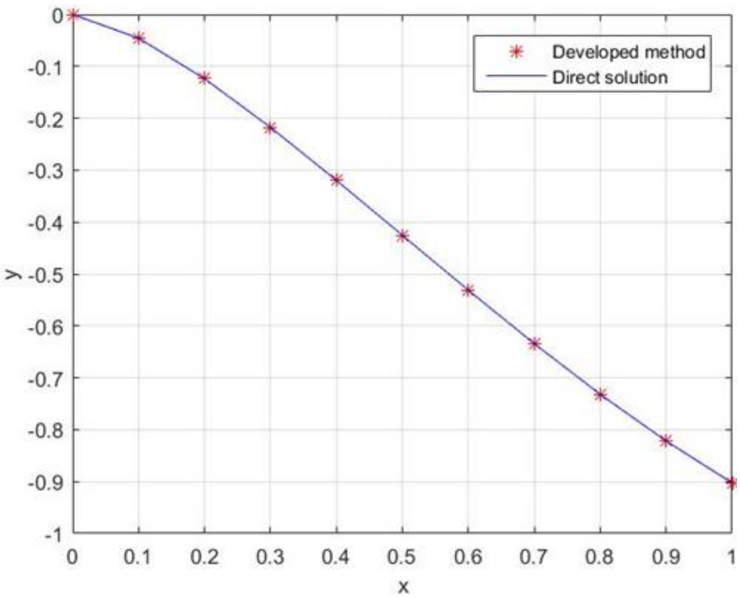

Figure (2): The approximate solution of Example (3) for some $\alpha = 0.5$

Table (2) Approximate solutions for the example(3)

x	$y(Direct\ solution)$	$y(Developed\ method)$	$Error$
0.0000000	0.0000000	0.0000000	0.0000000
0.1000000	-0.0456644	-0.0456736	0.0000092
0.2000000	-0.1237768	-0.1238017	0.0000249
0.3000000	-0.2175059	-0.2175496	0.0000438
0.4000000	-0.3196507	-0.3197150	0.0000643
0.5000000	-0.4254528	-0.4255384	0.0000856
0.6000000	-0.5313086	-0.5314156	0.0001069
0.7000000	-0.6342872	-0.6344149	0.0001276
0.8000000	-0.7318976	-0.7320448	0.0001473
0.9000000	-0.8219590	-0.8221244	0.0001654
1.0000000	-0.9025217	-0.9027033	0.0001816

5. conclusions:

In this paper, a new method has been successfully applied to solve heterogeneous fractal differential equations, which is the integration of the power series method and the method of approximating least squares.

The approximation is very close to the exact solution, so the results obtained show that the method is reliable and effective for solving a wide range of differential equations of the fractional order.

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