Intuitionistic Fuzzy Stability of Generalized Additive Set-Valued Functional Equation via Fixed Point Method

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Abstract

In this paper, we determine the intuitionistic fuzzy stability of generalized additive functional equation by utilizing the fixed point technique.

Keywords: Fixed point, Intuitionistic fuzzy normed space, t-norm, t-conorm, Jensen functional equation, Cauchy functional equation, generalized additive set-valued functional equation.

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh in 1965 [1]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments is made in the field of fuzzy topology [1-10]. The concept of fuzzy topology may have very important applications in quantum particle physics particularly in connections with both string and $\varepsilon\infty$ theory which were given and studied by Elnaschie [11-14]. One of the most important problems in fuzzy topology is to obtain an appropriate concept of intuitionistic fuzzy metric space. This problem has been investigated by Park [15-16]. They have introduced and studied a notion of intuitionistic fuzzy normed/metric space, and later Lael and Nourouzi [39] presented the modified version of the notion of intuitionistic fuzzy normed space. We recall it as follows.

Definition 1.1 [17]. *A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions:*

(a) $*$ is associative and commutative;

(b) $*$ is continuous;

(c) $a*1=a$ for all $a\in[0,1]$;

(d) $a*b\le c\le d$ whenever $a\le c$ and $b\le d$, for each $a, b, c, d\in[0,1]$.

Example 1.2 [16]. Two typical examples of continuous t-norm are $a*b=ab$ and $a*b=min(a;b)$.

Definition 1.3 [17]. *A binary operation $\blacklozenge:[0,1]\times[0,1]\to[0,1]$ is a continuous t-conorm if it satisfies the following conditions:*

(a) $\blacklozenge$ is associative and commutative;

(b) $\blacklozenge$ is continuous;

(c) $a\blacklozenge 0=a$ for all $a\in[0, 1]$;

(d) $a\blacklozenge b\le c\blacklozenge d$ whenever $a\le c$ and $b\le d$, for each $a, b, c, d\in[0, 1]$.

Example 1.4 [16]. *Two typical examples of continuous t-conorm are $a\blacklozenge b=min(a+b,1)$ and $a\blacklozenge b=max(a, b)$.*

Definition 1.4 [39]. *A 5-tuple $(X.M,N,*,\blacklozenge)$ is called an intuitionistic fuzzy normed space (IFNS) if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, $\blacklozenge$ a continuous t-conorm and M,N are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions for each $x,y,z\in X$ and $t, s>0$,*

(a) $M(x, t)=0$ for all non-positive t;

(b) $M(x, t)=1$ if and only if $x=0$;

(c) $M(cx, t)=M(x, t/|c|)$ for $c\neq 0$;

(d) $M(x,t)*M(y,s)\le M(x+y,t+s)$;

(e) $\lim_{t\to\infty} M(x,t)=1$ and $\lim_{t\to 0} M(x,t)=0$;

(f) $N(x, t)=1$ for all non-positive t;

(g) $N(x, t)=0$ if and only if $x=0$;

(h) $N(cx, t)=N(x, t/|c|)$ for $c\neq 0$;

(i) $N(x+y, t+s)\le N(x, t)\blacklozenge N(y, s)$;

(j) $\lim_{t\to\infty} N(x,t)=0$ and $\lim_{t\to 0} N(x,t)=1$.

Then *(M;N)* is called an intuitionistic fuzzy norm on X. For more details, we refer to [19-27].

For the following notions, we refer to [16].

Let *(X,M,N,*,♦)* be an IFNS. Then, a sequence $x=\{x_k\}$ is said to be intuitionistic fuzzy

(i) convergent to $\lambda\in X$ if $\lim M(x_k-\lambda, t)=1$ and $\lim N(x_k-\lambda, t)=0$ for all $t>0$. Thus we can said that x_k is convergence to λ in the intuitionistic fuzzy normed space (X,M,N,*,♦) for $k\to\infty$ and denote by (M, N)-lim $x=\lambda$.

(ii) Cauchy sequence if for all $t\in(0,\infty)$ and $p\in$, $\lim M(x_{k+p}-x_k, t)=1$ and $\lim N(x_{k+p}-x_k, t)=0$. Therefore, an intuitionistic fuzzy normed space *(X,M,N,*,♦)* is said to be complete if every intuitionistic fuzzy Cauchy sequence is intuitionistic fuzzy convergent in *(X,M,N,*,♦)*.

The Cauchy functional equation $f(x+y)=f(x)+f(y)$ is more of interest and an useful functional equation in real, complex Banach and fuzzy spaces. In 1978 T.M. Rassias proved this equation $\|f(x+y)-f(x)-f(y)\|\leq\epsilon(\|x\|p+\|y\|p)$ for an approximately additive mapping f from a normed vector space E into a Banach space E′:

Theorem 1.5 [18]. *Let f be an approximately additive mapping from a normed vector space E into a Banach space E', i.e. f satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\|\leq\epsilon(\|x\|p+\|y\|p)$$

for all $x,y\in E$, where ϵ and p are constants with $\epsilon>0$ and $0\leq p<1$. Then the mapping $\lambda:E\to E'$ defined by $\lambda(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$

is the unique additive mapping which satisfies $\|f(x)-\lambda(x)\|\leq\frac{2\varepsilon}{2-2^p}\|x\|^p$, $\forall x\in E$.

Remark 1.6 [28]. *Let (X,M,N,*,♦) be an intuitionistic fuzzy normed space with the conditions $M(x, t)>0$ and $N(x, t)<1$ implying $x=0$ for all $t\in$.. (1.1)*

Let $\|x\|_\alpha=\inf\{t>0:M(x, t)\geq\alpha$ and $N(x, t)\leq 1-\alpha\}$, for all $\alpha\in(0,1)$. Then $\{\|.\|_\alpha:\alpha\in(0,1)\}$ is an ascending family of norms on X in the sense that $\|x\|_\alpha\leq\|x\|_\beta$ for all $x\in X$ whenever $\alpha\leq\beta$. These norms are called α-norms on X corresponding to intuitionistic fuzzy norm (M, N).

In 2011, S.A. Mohiuddine, M. Cancan, H. Şevli determined a stability result concerning the Cauchy functional equation $f(x+y)=f(x)+f(y)$ in intuitionistic fuzzy normed spaces [29] by following theorem:

Theorem 1.7 [29]. *Let (M,N) and (M',N') satisfy (1.1). Assume also that (X,M,N) and (Y,M',N') are intuitionistic fuzzy normed space and intuitionistic fuzzy Banach space, respectively. Moreover, let $f:X\to Y$ be a q-approximately additive function in the sense that for some $0<q<1$*

$M'(f(x+y)-f(x)-f(y), t+s)\geq M(x, t^{-q})*M(y, s^{-q})$ and

$N'(f(x+y)-f(x)-f(y), t+s)\leq N(x, t^{-q})♦N(y, s^{-q})$, (1.2)

for all $x,y\in X$ and $t\in(0,\infty)$.

Then there exists a unique additive mapping $f^*:X\to Y$ such that

$$M'\left(f^*(x)-f(x),t\right)\le M\left(x,\left(\frac{\left(2-2^q\right)t}{2}\right)^{1/q}\right),$$
$$N'\left(f^*(x)-f(x),t\right)\ge N\left(x,\left(\frac{\left(2-2^q\right)t}{2}\right)^{1/q}\right). \qquad (1.3)$$

for all $x\in X$ and $t\in(0,\infty)$.

By utilizing Theorem 1.6 and Theorem 1.7, we obtain the intuitionistic fuzzy stability of generalized additive functional equations by fixed point technique in next section.

For more information about generalized additive functional equations, the readers can see [30-37].

2. Stability of generalized additive functional equation:

In 2013, Jang, Park and Cho defined a generalized additive set-valued functional equation, which is related to the following generalized additive functional equation [31]:

$$f\left(x_1+\ldots+x_n\right)=(n-1)\,f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}\right)+f\left(x_n\right) \qquad (2.1)$$

for a fixed $n\in$-{1}, and proved the Hyers-Ulam stability of the generalized additive set-valued functional equation.

In 2014, Sun Young Jang defined another version of generalized additive set-valued functional equations as

$$f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}+x_n\right)+f\left(\frac{x_1+\ldots+x_{n-2}+x_n}{n-1}+x_{n-1}\right)+\ldots+f\left(\frac{x_2+\ldots+x_n}{n-1}+x_1\right)$$
$$=2\left[f\left(x_1\right)+f\left(x_2\right)+\ldots+f\left(x_n\right)\right] \qquad (2.2)$$

which is related $f\left(x_1+\ldots+x_n\right)=(n-1)\,f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}\right)+f\left(x_n\right)$ for a fixed $n\in$-{1}, and they proved the Hyers-Ulam stability of the generalized additive set-valued functional equations by using of the fixed point method. [32]

The theory of set-valued functions has been much related to the Control theory and the mathematical economics. Now we intend to determine intuitionistic fuzzy stability for generalized additive functional equations via fixed point technique by using stability result of the Cauchy functional equation $f(x+y)=f(x)+f(y)$ in intuitionistic fuzzy normed spaces.

We rewrite Theorem 1.7 as follows:

Noation 2.1: *Let (X,M,N) be an intuitionistic fuzzy normed space, (Y,M',N') an intuitionistic fuzzy Banach space. Let also (M,N) and (M',N') satisfy (1.1). Suppose $f:X\to Y$ is a q-approximately additive function in the sense (for some $0<q<1$) and $y_i=\frac{\sum_{j=1,i\neq j}^{n}x_j}{n-1}$, for all $x_i,y_i\in X$, $i=1,\ldots,n$. Thus we have Cauchy functional equation $f(x_i+y_i)=f(x_i)+f(y_i)$ such that:*

$$M'\left(f\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1}+x_i\right)-f\left(x_i\right)-f\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1}\right),t+s\right)\geq M\left(x_i,t^{-q}\right)*M\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1},s^{-q}\right)$$ *and*

$$N'\left(f\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1}+x_i\right)-f\left(x_i\right)-f\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1}\right),t+s\right)\leq N\left(x_i,t^{-q}\right)\blacklozenge N\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1},s^{-q}\right). \quad (2.3)$$

for all $x_i,y_i \in X$, $i=1,\dots,n$ and $t\in(0,\infty)$.

Then there exists a unique additive mapping $f^*:X\to Y$ such that

$$M'\left(f^*\left(x_i\right)-f\left(x_i\right),t\right)\leq M\left(x_i,\left(\frac{\left(2-2^q\right)t}{2}\right)^{1/q}\right),$$

$$N'\left(f^*\left(x_i\right)-f\left(x_i\right),t\right)\geq N\left(x_i,\left(\frac{\left(2-2^q\right)t}{2}\right)^{1/q}\right). \quad (2.4)$$

for all $x_i\in X$, $i=1,\dots,n$ and $t\in(0,\infty)$.

In this section, we deal with the stability problem via the fixed point method in intuitionistic fuzzy norm space. Before proceeding further, we should recall the following results related to the concept of fixed point.

Definition 2.2:*Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies*

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,z)\leq d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Note that the distinction between the generalized metric and the usual metric is that the range of the former includes the infinity.

Theorem 2.3 [29]. *(Banach's Contraction Principle). Let (χ, d) be a complete generalized metric space and consider a mapping $J:\chi\to\chi$ as a strictly contractive mapping, that is $d(Jx, Jy)\leq Ld(x,y)$, $\forall x,y\in\chi$ for some (Lipschitz constant) $L<1$. Then*

(i) The mapping J has one and only one fixed point $x^*=J(x^*)$;

(ii) The fixed point $x^*$is globally attractive, that is $\lim J^n x=x^*, n\to\infty$ for any starting point $x\in\chi$;

(iii) One has the following estimation inequalities for all $x\in\chi$ and $n\geq 0$:

$$d\left(J^n x,\ x^*\right)\ \leq\ L^n.d\left(x,\ x^*\right)\ \ (2.5)$$

$$d\left(J^n x,\ x^*\right)\ \leq\left(\frac{1}{1-L}\right)d\left(J^n x,\ J^{n+1}x\right)\ \ (2.6)$$

$$d\left(x,\ x^*\right)\ \leq\left(\frac{1}{1-L}\right)d\left(x,\ Jx\right)\ \ (2.7)$$

Theorem 2.4 [38]. *(The Alternative of Fixed Point). Suppose we are given a complete generalized metric space (χ, d) and a strictly contractive mapping $J:\chi\to\chi$, with Lipschitz constant L. Then, for each given element $x\in\chi$, either*

$$d(J^n x,\ J^{n+1}x)=+\infty,\ \forall n\geq 0 \quad (2.8)$$

or

$$d(J^n x,\ J^{n+1}x)<+\infty\ \forall n\geq \eta^\circ \quad (2.9)$$

for some natural number η°. Moreover, if the second alternative holds then

(i) The sequence ($J^n x$) is convergent to a fixed point x^* of J;

(ii) x^* is the unique fixed point of J in the set $\bar{X}=\{\bar{x}\in X,\ d(J^{\eta^\circ}x,\ \bar{x})<+\infty\}$

(iii) $d(\bar{x},\ x^*)\ \leq \left(\dfrac{1}{1-L}\right) d(\bar{x},\ J\bar{x})\ \forall \bar{x}\in\bar{X}$ *(2.10)*

Theorem 2.5. *Let (M, N) and (M′, N′) satisfy (1.1) and (X, M, N) and (Y,M′, N′) be two intuitionistic fuzzy normed space and intuitionistic fuzzy Banach space, respectively. Suppose that $f:X\to Y$ is a q-approximately additive function in the sense that for some $0<q<1$*

$$M'\left(f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}+x_n\right)+f\left(\frac{x_1+\ldots+x_{n-2}+x_n}{n-1}+x_{n-1}\right)+\ldots+f\left(\frac{x_2+\ldots+x_n}{n-1}+x_1\right)-2\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right],t\right)$$
$$\geq *\prod_{i=1}^{n}\left(M\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*M\left(\frac{\sum_{j=1,j\neq i}^{n}x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)\right)$$

and

$$N'\left(f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}+x_n\right)+f\left(\frac{x_1+\ldots+x_{n-2}+x_n}{n-1}+x_{n-1}\right)+\ldots+f\left(\frac{x_2+\ldots+x_n}{n-1}+x_1\right)-2\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right],t\right)$$
$$\leq \blacklozenge\prod_{i=1}^{n}\left(N\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*N\left(\frac{\sum_{j=1,j\neq i}^{n}x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)\right)$$

for all $x_i\in X$, i=1,...,n and $t\left(=\sum_{j=1}^{n}t_j\right)\in(0,\infty)$.

Then there exists a unique additive mapping $f^*:X\to Y$ such that

$$M\left(f^*(x)-f(x),\ t\right)\geq *\prod_{i=1}^{n}\left(M\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*M\left(\frac{\sum_{j=1,j\neq i}^{n}x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)\right)$$

and

$$N\left(f^*(x)-f(x),t\right)\leq \blacklozenge\prod_{i=1}^{n}\left(N\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*N\left(\frac{\sum_{j=1,j\neq i}^{n}x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)\right)$$

for all $x\left(=\sum_{j=1}^{n}x_j\right)\in X$ *and all* $t\left(=\sum_{j=1}^{n}t_j\right)\in(0,\infty)$.

Proof. By Theorem 1.7, we will have

$$M'\left(f\left(\frac{\sum_{j=1,i\neq j}^{n}x_j}{n-1}+(x_i)\right)-f(x_i)-f\left(\frac{\sum_{j=1,i\neq j}^{n}x_j}{n-1}\right),\frac{t_i}{2}+\frac{t_i}{2}\right)\geq M\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*M\left(\frac{\sum_{j=1,i\neq j}^{n}x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)$$

for all $x_i\in X$, i=1,...,n and $t\in(0,\infty)$.

Therefore by definition of the intuitionistic fuzzy normed space (X, M, N), we have:

$$M'\left(\begin{array}{l} f\left(\frac{\sum_{j=2}^{n} x_j}{n-1}+(x_1)\right)+f\left(\frac{\sum_{j=1,j\neq 2}^{n} x_j}{n-1}+(x_2)\right)+....+f\left(\frac{\sum_{j=1,j\neq n}^{n} x_j}{n-1}+(x_n)\right)-f(x_1)-f(x_2)-...-f(x_n) \\ -f\left(\frac{\sum_{j=2}^{n} x_j}{n-1}\right)-f\left(\frac{\sum_{j=1,j\neq 2}^{n} x_j}{n-1}\right)-...-f\left(\frac{\sum_{j=1,j\neq n}^{n} x_j}{n-1}\right),t_1+t_2+...+t_n \end{array}\right)$$

$$\geq M\left(x_1,\left(\frac{t_1}{2}\right)^{1/q}\right)*M\left(\frac{\sum_{j=2}^{n} x_j}{n-1},\left(\frac{t_1}{2}\right)^{1/q}\right)*M\left(x_2,\left(\frac{t_2}{2}\right)^{1/q}\right)*M\left(\frac{\sum_{j=1,j\neq 2}^{n} x_j}{n-1},\left(\frac{t_2}{2}\right)^{1/q}\right)*...$$

$$*M\left(x_n,\left(\frac{t_n}{2}\right)^{1/q}\right)*M\left(\frac{\sum_{j=1,j\neq n}^{n} x_j}{n-1},\left(\frac{t_n}{2}\right)^{1/q}\right)$$

And also

$$N'\left(f\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1}+x_i\right)-f(x_i)-f\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1}\right),t_i\right)\leq N\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)\blacklozenge N\left(\frac{\sum_{j=1,i\neq j}^{n} x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right).$$

for all $x_i \in X$, i=1,...,n and $t\in(0,\infty)$.

Hence by definition of the intuitionistic fuzzy Banach space (Y,M′, N′), we observe that

$$N'\left(\begin{array}{l} f\left(\frac{\sum_{j=2}^{n} x_j}{n-1}+(x_1)\right)+f\left(\frac{\sum_{j=1,j\neq 2}^{n} x_j}{n-1}+(x_2)\right)+....+f\left(\frac{\sum_{j=1,j\neq n}^{n} x_j}{n-1}+(x_n)\right)-f(x_1)-f(x_2)-...-f(x_n) \\ -f\left(\frac{\sum_{j=2}^{n} x_j}{n-1}\right)-f\left(\frac{\sum_{j=1,j\neq 2}^{n} x_j}{n-1}\right)-...-f\left(\frac{\sum_{j=1,j\neq n}^{n} x_j}{n-1}\right),t_1+t_2+...+t_n \end{array}\right)$$

$$\leq N\left(x_1,\left(\frac{t_1}{2}\right)^{1/q}\right)\blacklozenge N\left(\frac{\sum_{j=2}^{n} x_j}{n-1},\left(\frac{t_1}{2}\right)^{1/q}\right)\blacklozenge N\left(x_2,\left(\frac{t_2}{2}\right)^{1/q}\right)\blacklozenge N\left(\frac{\sum_{j=1,j\neq 2}^{n} x_j}{n-1},\left(\frac{t_2}{2}\right)^{1/q}\right)\blacklozenge...$$

$$\blacklozenge N\left(x_n,\left(\frac{t_n}{2}\right)^{1/q}\right)\blacklozenge N\left(\frac{\sum_{j=1,j\neq n}^{n} x_j}{n-1},\left(\frac{t_n}{2}\right)^{1/q}\right)$$

Suppose now

$$A(x,\ t)=*\prod_{i=1}^{n}\left(M\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*M\left(\frac{\sum_{j=1,j\neq i}^{n} x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)\right)$$

for all $x\left(=\sum_{j=1}^{n} x_j\right)\in X$ and all $t\left(=\sum_{j=1}^{n} t_j\right)\in(0,\infty)$

$$B(x,\ t)=\blacklozenge\prod_{i=1}^{n}\left(N\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*N\left(\frac{\sum_{j=1,j\neq i}^{n} x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)\right)$$

for all $x\left(=\sum_{j=1}^{n} x_j\right)\in X$ and all $t\left(=\sum_{j=1}^{n} t_j\right)\in(0,\infty)$

Now we consider the set $E:=\{g:X\to Y,\ g(0)=0\}$ together with the mapping d_E defined on E×E by $d_E(g_1,g_2)=\inf\{a\geq 0: M(g_1(x)-g_2(x),\ at)\geq A(x,\ t) \text{ and } N(g_1(x)-g_2(x),\ at)\leq B(x,\ t)\}$, (2.11) for all $x\left(=\sum_{j=1}^{n} x_j\right)\in X$ and all $t\left(=\sum_{j=1}^{n} t_j\right)\in(0,\infty)$. It is easy to see that $d_E(g_1,g_2)$ is a complete generalized metric space.

For a→∞:A(x,at)=1≥ $*\prod_{i=1}^{n}\left(M\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*M\left(\frac{\sum_{j=1,j\neq i}^{n} x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)\right)$

since 1≥M(x,t)>0 for all x∈X, t∈(0,∞).

And B(x,at)=0≤ $*\prod_{i=1}^{n}\left(N\left(x_i,\left(\frac{t_i}{2}\right)^{1/q}\right)*N\left(\frac{\sum_{j=1,j\neq i}^{n} x_j}{n-1},\left(\frac{t_i}{2}\right)^{1/q}\right)\right)$

since 1>N(x,t)≥0 for all x∈X, t∈(0,∞).

Let $g_1,g_2, g_3\in E$, $d_E(g, h)<a_1$ and $d_E(h, k)<a_2$. Thus for all x∈X, t∈(0,∞):

$M(g_1(x)-g_2(x), a_1t)\geq A(x, t), M(g_2(x)-g_3(x), a_2t)\geq A(x, t);$

and

$N(g_1(x)-g_2(x), a_1t)\leq B(x, t), N(g_2(x)-g_3(x), a_2t)\leq B(x, t).$

These imply that

$M(g_1(x)-g_3(x), (a_1+a_2)t)\geq M(g_1 (x)-g_2(x), a_1t)*M(g_2(x)-g_3(x), a_2t)\geq A(x, t),$

and

$N(g_1(x)-g_3(x), (a_1+a_2)t)\leq N(g_1 (x)-g_2(x), a_1t)\blacklozenge N(g_2(x)-g_3(x), a_2t)\leq B(x, t),$

for each x∈X, t∈(0,∞). And the triangle inequality for d_E will conclude on based definition of d_E and $d_E(g_1,g_2)\leq a_1+a_2$. The rest of the conditions follow directly from the definition.

Now, we define the linear mapping $J:E\to E$ such that

$Jg(x)=\frac{1}{n+1}g\left((n+1)x\right)$ for fixed integer number n.

Obviously, J is a strictly contractive self-mapping of E with the Lipschitz constant $(n+1)^{-1}<1$.

Suppose now $g_1,g_2\in E$ be given such that $d_E(g_1,g_2)<a$ (for all a∈(0,∞)). So

$M(g_1 (x)-g_2(x), at)\geq A(x, t)$, and $N(g_1 (x)-g_2(x), at)\leq B(x, t),$

for all x∈X and t>0, n+1>a>0. It follows from (2.11) that

$d_E(Jg_1,Jg_2)=\inf\{a\geq 0:M(Jg_1(x)-Jg_2(x), at)\geq A(x, t) \text{ and } N(Jg_1(x)-Jg_2(x), at)\leq B(x, t)\}$

$$=\inf\left\{\begin{array}{l} a\geq 0: M\left(\frac{1}{n+1}g_1\left((n+1)x\right)-\frac{1}{n+1}g_2\left((n+1)x\right),at\right)\geq A\left((n+1)x,t\right)\ \& \\ N\left(\frac{1}{n+1}g_1\left((n+1)x\right)-\frac{1}{n+1}g_2\left((n+1)x\right),at\right)\leq B\left((n+1)x,t\right)\end{array}\right\}$$

$$=\inf\left\{a\geq 0: M\left(g_1\left((n+1)x\right)-g_2\left((n+1)x\right),(n+1)at\right)\geq A\left(x,t\right)\ \&\ N\left(g_1\left((n+1)x\right)-g_2\left((n+1)x\right),(n+1)at\right)\leq B\left(x,t\right)\right\}$$

$$a\geq 0:\ M\left(g_1(x)-g_2(x),at\right)\geq A\left(x,\frac{t}{(n+1)}\right)\ \&\ N\left(g_1(x)-g_2(x),at\right)\leq B\left(x,\frac{t}{(n+1)}\right)\qquad(2.12)$$

$$M\left(Jg_1(x)-Jg_2(x),\frac{a}{n+1}t\right)=M\left(\frac{1}{n+1}g_1\left((n+1)x\right)-\frac{1}{n+1}g_2\left((n+1)x\right),\frac{a}{n+1}t\right)= \qquad (2.13)$$
$$M\left(g_1\left((n+1)x\right)-g_2\left((n+1)x\right),at\right)\geq A\left((n+1)x,at\right).$$

and

$$N\left(Jg_1(x)-Jg_2(x),\frac{a}{n+1}t\right)=N\left(\frac{1}{n+1}g_1\left((n+1)x\right)-\frac{1}{n+1}g_2\left((n+1)x\right),\frac{a}{n+1}t\right)=$$
$$N\left(g_1\left((n+1)x\right)-g_2\left((n+1)x\right),at\right)\le B\left((n+1)x,at\right). \tag{2.14}$$

for all x∈X and t>0, n+1>a>0. It follows from (2.11), (2.12), (2.13) and (2.14) that

$$\left.\begin{array}{l} M\left(Jg_1(x)-Jg_2(x),\frac{a}{n+1}t\right)\ge A\left((n+1)x,at\right) \\ N\left(Jg_1(x)-Jg_2(x),\frac{a}{n+1}t\right)\le B\left((n+1)x,at\right) \\ d_E(g_1,g_2)\ <\ a\Rightarrow d_E(Jg_1,Jg_2)<\frac{a}{n+1}<1 \end{array}\right\}\Rightarrow d_E(Jg_1,Jg_2)<\frac{a}{n+1}d_E(g_1,g_2).$$

On the other hands, from $Jf(x)=\frac{1}{n+1}f\left((n+1)x\right)$ and (2.11), we have

$d_E(Jf,f)=inf\{a\ge0:M(Jf(x)-f(x), at)\ge A(x, t)\ \&\ N(Jf(x)-f(x), at)\le B(x, t)\}$

$$= inf\left\{a\ge0:M\left(\frac{1}{n+1}f\left((n+1)x\right)-f\left(x\right),at\right)\ge A\left(x,t\right)\&N\left(\frac{1}{n+1}f\left((n+1)x\right)-f\left(x\right),at\right)\le B\left(x,t\right)\right\}$$

Now by Theorem 2.1, it follows that $d_E(f,Jf)\le1$.

Using the fixed point alternative we deduce the existence of a fixed point of J, that is, the existence *a unique generalized Cauchy-Jensen type additive set-valued mapping* f*:X→Y such that $f^*(x)=\frac{1}{n}f^*\left(nx\right)$ or nf*(x)=f*(nx), for all n∈.

Now by Notation 1, we can see that *f*(x+y)=f*(x)+f*(y)* and thus for $x=\sum_{j=1}^{n}x_j\in X$ and $y_i=\frac{\sum_{j=1,i\ne j}^{n}x_j}{n-1}$.

$$f^*\left(\frac{x_1+...+x_{n-1}}{n-1}+x_n\right)+f^*\left(\frac{x_1+...+x_{n-2}+x_n}{n-1}+x_{n-1}\right)+...+f^*\left(\frac{x_2+...+x_n}{n-1}+x_1\right)$$
$$=f^*\left(x_1\right)+...+f^*\left(x_n\right)+f^*\left(\frac{x_1+...+x_{n-1}}{n-1}\right)+f^*\left(\frac{x_1+...+x_{n-2}+x_n}{n-1}\right)+...+f^*\left(\frac{x_2+...+x_n}{n-1}\right)$$

So

$$f^*\left(\frac{x_1+...+x_{n-1}+\left(n-1\right)x_n}{n-1}\right)+f^*\left(\frac{x_1+...+x_{n-2}+\left(n-1\right)x_{n-1}+x_n}{n-1}\right)+...+f^*\left(\frac{\left(n-1\right)x_1+x_2+...+x_n}{n-1}\right)$$
$$=f^*\left(x_1\right)+...+f^*\left(x_n\right)+f^*\left(\frac{x_1+...+x_{n-1}}{n-1}\right)+f^*\left(\frac{x_1+...+x_{n-2}+x_n}{n-1}\right)+...+f^*\left(\frac{x_2+...+x_n}{n-1}\right)$$
$$=\sum_{i=1}^{n}f^*\left(x_i\right)+\frac{1}{\left(n-1\right)}\sum_{j=1,j\ne i}^{n}f^*\left(x_1+...+x_{i-1}+x_{i+1}+...+x_n\right)$$

Therefore,

$$\left[\frac{\sum_{i=1}^{n}f^*\left(\left(n-1\right)x_i+\sum_{j=1,j\ne i}^{n}x_j\right)}{\left(n-1\right)}\right]=\sum_{i=1}^{n}f^*\left(x_i\right)+\frac{1}{\left(n-1\right)}\sum_{i=1}^{n}\sum_{j=1,j\ne i}^{n}f^*\left(x_j\right)$$

$$=\sum_{i=1}^{n} f^{*}\left(x_i\right)+\frac{1}{(n-1)}\sum_{i=1}^{n}\left[(n-1) f^{*}\left(x_i\right)\right]$$

$$=\sum_{i=1}^{n} f^{*}\left(x_i\right)+\sum_{i=1}^{n} f^{*}\left(x_i\right)=2\left[f^{*}\left(x_1\right)+f^{*}\left(x_2\right)+\ldots+f^{*}\left(x_n\right)\right]$$

These imply that

$$f^{*}\left(\frac{x_1+\ldots+x_{n-1}}{n-1}+x_n\right)+f^{*}\left(\frac{x_1+\ldots+x_{n-2}+x_n}{n-1}+x_{n-1}\right)+\ldots$$

$$+f^{*}\left(\frac{x_2+\ldots+x_n}{n-1}+x_1\right)=2\left[f^{*}\left(x_1\right)+f^{*}\left(x_2\right)+\ldots+f^{*}\left(x_n\right)\right]$$

for all x∈X. Moreover, we have $d_E(J^m f, f^*)\to 0$, for $m\to\infty$:which implies

$$(M,N)-\lim_{m\to\infty}\left(\frac{f\left((n+1)^m x\right)}{(n+1)^m}\right)=f^{*}(x)$$

for all x∈X. And

$d_E(f^*,f)=\inf\{a\geq 0: M(f^*(x)-f(x), at)\geq A(x, t) \text{ and } N(f^*(x)-f(x), at)\leq B(x, t)\}$,

$$d_E\left(f^{*}, f\right)\leq \frac{1}{1-\left(\frac{1}{n+1}\right)} d_E(Jf, f)=\left(\frac{n+1}{n}\right) d_E(Jf, f)\leq \frac{n+1}{n}.$$

This implies that

$$M\left(f^{*}(x)-f(x),\left(\frac{n+1}{n}\right)t\right)\geq A(x, t) \quad N\left(f^{*}(x)-f(x),\left(\frac{n+1}{n}\right)t\right)\leq B(x, t)$$

and

Thus

$$M\left(f^{*}(x)-f(x), t\right)\geq A\left(x,\left(\frac{n}{n+1}\right)t\right) \quad N\left(f^{*}(x)-f(x), t\right)\leq B\left(x,\left(\frac{n}{n+1}\right)t\right)$$

and

for all x∈X, n=1,2,... and t∈(0,∞).■

Conclusion:

In this paper, by using the fixed point alternative, we proved the stability of the Hyers-Ulam-Rassias type theorem in the intuitionistic fuzzy stability space.

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