General Solution of Telegraph Equation Using Aboodh Transform

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Abstract

This paper describes using the Aboodh integral transform in solving one-dimensional second-order hyperbolic telegraph equations with satisfying initial conditions by solving some telegraph equations with specific initial conditions and then generalizing the solution to the telegraph equation using the Aboodh integral transform.

1. Introduction

Since the 19th century the importance of the communication field has been raised, and many researches have been performed to improve the different aspects of the communication. Data communication is performed via transmission media, the transmission media is either wired or wireless, the wired communications is the earliest form of communication and the telegraph system is the root of the wired communications, [1].

The telegraph equation had been introduced by Oliver Heaviside in 1876, it describes the reflection of the electromagnetic waves in the transmission medium. The telegraph equation is a linear partial differential equation that describe the relationship between the transmitted voltage and current with time and distance. Although telegraph equation originally proposed to be applied into telegraph wires, but it can be applied into transmission lines with all frequencies, and that gave the telegraph equation its immense importance in the communications field, [2-4].

Integral transforms represent a mathematical method that can be used to solve differential equations, many integral transforms have been extensively used to solve differential equations, such as Laplace, Elzaki, Mellin, Mohand, Mahgoub and Al-Tememe [5-8]. In (2014) Khalid Suliman Aboodh had proposed an integral transform that showed the proprieties of Mohanad transform, [6], and since then Aboodh integral transform has been used in solving the differential equations of some applications [9][10], however due to its novelty it hasn't been utilized in many important fields, in this paper Aboodh integral transform is going to be applied on the telegraph differential equation as an application of the communications field.

2. The Telegram Equation

The importance of telegraph equation leads to the proposal of many methods to effectively solve it [3,11-14]. The general form of telegram equation is:

$$\frac{\partial^2 u}{\partial t^2} + (\alpha + \beta)\frac{\partial u}{\partial t} + \alpha\beta u = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Where $u(x,t)$ can be voltage or current through the wire at position x and time t, $\alpha = \frac{G}{C}$, $\beta = \frac{R}{L}$ and $C^2 = \frac{1}{LC}$, where G is conductance of resistor, R is resistance of resistor, L is inductance of coil, and C is capacitance of capacitor. The telegraph equation components are shown in figure 1, [14].
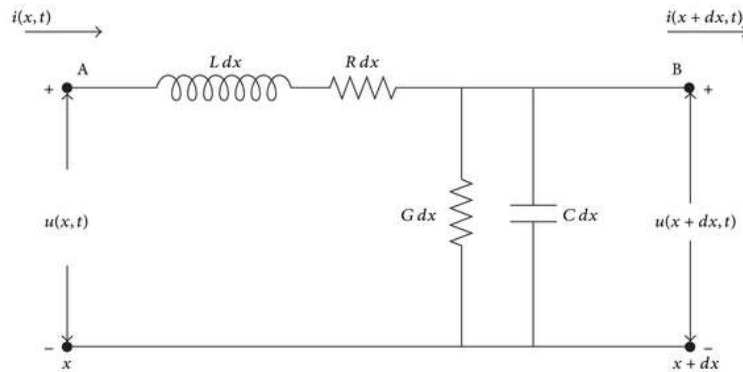

Figure 1: Telegraph Transmission line [3]

3.   Aboodh Transform Basic Definitions and Principles
 In this section some definitions and properties of Aboodh transformation are demonstrated.

3.1.   Aboodh transform definition [5]
Aboodh transform defined for a function of exponential order in the A set as: $A = \{f(t): \exists\ M, k_1, k_2 > 0, |f(t)| < M\ e^{-vt}$.

$f(t)$ is a given function in the set A, M is a finite number and k_1, k_2 could be finite or infinite.

Aboodh transform denoted by $A(.)$ is defined by: $A[f(t)] = \frac{1}{v}\int_0^\infty e^{-vt} f(t)dt$.

The variable v in this transform is used to factor the variable t in the argument of the function f.

3.2.   Aboodh transform for some functions [5]
- $A(1) = \frac{1}{v^n}$
- $A(t^n) = \frac{n!}{v^{n+2}}$, $n \in \mathbb{Z}^+$
- $A(e^{at}) = \frac{1}{v^2-av}$, a is constant
- $A(\sin(at)) = \frac{a}{v(v^2+a^2)}$
- $A(\cos(at)) = \frac{1}{v^2+a^2}$
- $A(\sinh(at)) = \frac{a}{v(v^2-a^2)}$
- $A(\cosh(at)) = \frac{1}{v^2-a^2}$

3.3.   Aboodh transform of derivative functions [5]
     Let $K(v)$ be the Aboodh transform where $[A(f(t) = K(v)]$, then:
- $A[f'(t)] = vK(v) - \frac{f(0)}{v}$.
- $A[f''(t)] = v^2K(v) - \frac{f'(0)}{v} - f(0)$.
- $A[f^{(n)}(t)] = v^nK(v) - \sum_{k=0}^{n-1}\frac{f^K(0)}{v^{2-n+k}}$.

3.4. *Aboodh transform of partial derivative functions [5]*

Let $K(x,v)$ be the Aboodh transform where $[A(f(x,t) = K(x,v)]$, then:

- $A\left[\frac{\partial f}{\partial x}(x,t)\right] = vK(x,v) - \frac{f(x,0)}{v}$ and $A\left[\frac{\partial f}{\partial x}\right] = \frac{d}{dx}[K(x,v)]$ and $A\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}[K(x,v)]$.

- $A\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right] = v^2 K(x,v) - \frac{1}{v}\frac{\partial f}{\partial t}(x,0) - f(x,0)$, it is possible to extend the results into the nth partial derivative by using mathematical induction.

4. Solving Telegraph Equations Using Aboodh Transform

Aboodh transform can be used to solve the differential equation of telegraph in an efficient manner, to prove the capability of this transform in solving the telegraph equation, two telegraph equations with specified initial conditions are going to be solve using Aboodh integral transform, then a generalize solution for the telegraph equation using Aboodh integral transform will be represented.

Telegraph equation 1: For the linear telegraph equation: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u$, with initial conditions:

$$u(x,0) = e^x \ , \ and \ u_t(x,0) = -2e^x .$$

To solve this telegraph equation, Aboodh integral transform is going to be applied as follow:

$$A\left[\frac{\partial^2 u}{\partial x^2}\right] = A\left[\frac{\partial^2 u}{\partial t^2}\right] + 2A\left[\frac{\partial u}{\partial t}\right] + A[u(x,t)] .$$

$$K''(x,v) - \left[v^2 K(x,v) - \frac{1}{v}\frac{\partial u}{\partial t}(x,0) - u(x,0)\right] - 2\left[vK(x,v) - \frac{u(x,0)}{v}\right] - K(x,v) = 0 .$$

By applying the initial conditions to the Aboodh integral transform of the equation:

$$K''(x,v) - v^2 K(x,v) - \frac{2e^x}{v} + e^x - 2vK(x,v) + \frac{2e^x}{v} - K(x,v) = 0 .$$

$$K''(x,v) - (v^2 + 2v + 1)K(x,v) = -e^x ,$$

$$\frac{1}{v^2+2v+1}K''(x,v) - K(x,v) = \frac{-e^x}{v^2+2v+1} ,$$

$$\frac{1}{(v+1)^2}K''(x,v) - K(x,v) = \frac{-e^x}{(v+1)^2} ,$$

$$K(x,v) = \frac{\frac{-e^x}{(v+1)^2}}{\frac{1}{(v+1)^2}(D^2-1)} .$$

Then, $K(x,v) = \frac{-e^x}{1-(v^2+2v+1)}$,

And, $K(x,v) = \frac{e^x}{v^2+2v}$.

The mathematical form of the function $u(x,t)$ could be obtained using inverse Aboodh integral transform as:
$u(x,t) = e^x . e^{-2t} = e^{x-2t}$

Telegraph equation 2: Considering the linear telegraph equation: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 4\frac{\partial u}{\partial t} + 4u$, with initial conditions: $u(x,0) = 1 + e^{2x} \ , \ and \ u_t(x,0) = -2$.

To solve this telegraph equation, Aboodh integral transform is going to be applied as follow:

$$A[u_{xx}] - A[u_{tt}] - 4A[u_t] - 4A[u] = 0 ,$$

$$K''(x,v) - \left[v^2 K(x,v) - \frac{1}{v}u_t(x,0) - u(x,0)\right] - 4\left[vK(x,v) - \frac{u(x,0)}{v}\right] - 4K(x,v) = 0.$$

By applying the initial conditions to the Aboodh integral transform of the equation:

$$K''(x,v) - v^2 K(x,v) - \frac{2}{v} + 1 + e^{2x} - 4vK(x,v) + 4\frac{(1+e^{2x})}{v} - 4K(x,v) = 0,$$

$$K''(x,v) - (v^2 + 4v + 4)K(x,v) = \frac{2}{v} - 1 - e^{2x} - \frac{4}{v} - \frac{4e^{2x}}{v} ,$$

$$K''(x,v) - (v+2)^2 K(x,v) = \frac{2-v-ve^{2x}-4-4e^{2x}}{v} ,$$

$$\frac{1}{(v+2)^2}K''(x,v) - K(x,v) = \frac{-2-4e^{2x}}{v(v+2)^2} - \frac{v(1-e^{2x})}{v(v+2)^2} ,$$

$$K(x,v) = \frac{\frac{-2-4e^{2x}}{v(v+2)^2}}{\frac{1}{(v+2)^2}D^2-1} - \frac{\frac{1+e^{2x}}{(v+2)^2}}{\frac{1}{(v+2)^2}D^2-1} \ ,$$

The mathematical form of the function $u(x,t)$ could be obtained using inverse Aboodh integral transform as:
$u(x,t) = e^{2x} + e^{-2t}$.

5. Generalizing The Solution of Telegraph Equation by Using Aboodh Integral Transform

By taking the previous equations that had been solved using Aboodh transform as examples, it is possible to solve the telegraph equation with any initial conditions using Aboodh integral transform.

For the One-dimensional second-order hyperbolic telegraph equation with satisfied initial conditions: $\frac{\partial^2 u}{\partial t^2} +$

$$(\alpha + \beta)\frac{\partial u}{\partial t} + \alpha\beta u = c^2\frac{\partial^2 u}{\partial x^2},$$
$$c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} - (\alpha + \beta)\frac{\partial u}{\partial t} - \alpha\beta u = 0 \ .$$

It is possible to solve this equation using Aboodh integral transform as follows:
$$C^2 A[u_{xx}] - A[u_{tt}] - (\alpha + \beta)A[u_t] - \alpha\beta A[u] = 0 \ ,$$
$$C^2 K''(x,v) - \left[v^2 K(x,v) - \frac{1}{v}u_t(x,0)\right] - (\alpha + \beta)\left[vK(x,v) - \frac{u(x,0)}{v}\right] - \alpha\beta K(x,v) = 0 \ .$$

By applying the following initial conditions to the Aboodh integral transform of the telegraph equation:
$$u(x,0) = g(x) \ , \ u_t(x,0) = a \ \ ,\text{where a is constant} \ .$$

$$C^2 K''(x,v) - v^2 K(x,v) + \frac{a}{v} + g(x) - (\alpha + \beta)vK(x,v) + \frac{(\alpha+\beta)}{v}g(x) - \alpha\beta K(x,v) = 0 \ .$$

$$C^2 K''(x,v) - (v^2 + (\alpha + \beta)v - \alpha\beta)K(x,v) = \frac{-a}{v} - g(x) - \frac{(\alpha + \beta)}{v}g(x)$$

$$\frac{c^2}{v^2+(\alpha+\beta)v-\alpha\beta}K''(x,v) - K(x,v) = \frac{-a}{v(v^2+(\alpha+\beta)v-\alpha\beta)} - \frac{g(x)}{(v^2+(\alpha+\beta)v-\alpha\beta)} - \frac{(\alpha+\beta)g(x)}{v(v^2+(\alpha+\beta)v-\alpha\beta)} \ \ .$$

$a, g(x), \alpha$ and β are constants, the Aboodh transform for the telegraph equation is:

$$K(x,v) = \frac{a}{v(v^2+(\alpha+\beta)v-\alpha\beta)} + \frac{g(x)}{(v^2+(\alpha+\beta)v-\alpha\beta)} + \frac{(\alpha+\beta)g(x)}{v(v^2+(\alpha+\beta)v-\alpha\beta)} + \frac{c^2}{v^2+(\alpha+\beta)v-\alpha\beta}K''(x,v) \ \ .$$

The mathematical form of the function $u(x,t)$ could be obtained using inverse Aboodh integral transform.

6. Conclusions

Solving telegraph equations with specific initial conditions using the Aboodh integral transform proved the ability of the transform to solve this kind of equation. Although many integral transformations can solve the telegraph equation, the Aboodh integral transform provides a simple mathematical method to do so. The simplicity of the Aboodh transform comes from the direct substitution method that the transform provides and its lack of use of integrals and derivatives. That propriety gave the Aboodh integral transform a vast advantage over the more popular integral transforms.

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