Minimal Prime Ideals in Generalized Almost Distributive Fuzzy Lattices

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Abstract
The Minimal Prime Ideals (MPIs) in Generalized Almost Distributive Fuzzy Lattices are presented in this article (GADFL). In GADFL, we have also deduced certain properties and characteristic theorems of MPIs. Further, we have also derived the following theorems: let $I_f$ and $J_f$ be two ideals of GADFL $L(R_f, A_f)$. Then $I_f \land J_f$ is a MPI belonging to both $I_f$ and $J_f$ and finally, every ideal of GADFL $L(R_f, A_f)$ is the union of all MPIs containing it.

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1. Introduction:
G.C. Rao, Ravi Kumar Bandaru, and N. Rafi proposed the definition of Generalized Almost Distributive Lattices (GADFL) as a generalisation of Almost Distributive Lattices (ADLs). L.A. Zadeh [7] developed the notion of a fuzzy set in 1965. According to L.A. Zadeh [8], a fuzzy ordering is a transitive fuzzy relation that is a generalisation of the concept of ordering. N. Ajmal and K.V. Thomas [1] developed a fuzzy lattice as a fuzzy algebra in 1994, and fuzzy sub lattices were established in 1995. In 2009, I. Chon [4] developed a unique notion of fuzzy lattices and examined the level sets of fuzzy lattices based on fuzzy order theory. He also created the concepts of distributive and modular fuzzy lattices, as well as several basic fuzzy lattice properties. Berhanu et al. [2] proposed ADFLs as a generalisation of DFLs, and used I. Chon's fuzzy partial order relations and fuzzy lattices to characterise some elements of an ADL. Berhanu and Yohannes [3] define GADFLs as a generalisation of ADFLs.
In this article, we are presented the MPIs in GADFL. Further we have derived some properties and characteristic theorems of MPIs in GADFL. Also, we have derived the
theorems are, let $I$ and $J$ be two ideals of GADFL $L(R, A)$. Then $I \land J$ is a MPI belonging to both $I$ and $J$ and finally, every ideal of GADFL $L(R, A)$ is the union of all MPIs containing it.

2. Preliminaries:
A few fundamental definitions are discussed.

**Definition 2.1.** Let $(R, \lor, \land)$ be a fuzzy poset and $(R, A)$ be an algebra type (2,2). If $(R, A)$ meets the following axioms, we call it a GADFL.

1. $A((a \land b) \land c, a \land (b \land c)) = A(a \land (b \land c), (a \land b) \land c) = 1$;
2. $A(a \land (b \lor c), (a \land b) \lor (a \land c)) = A((a \land b) \lor (a \land c), a \land (b \lor c)) = 1$;
3. $A(a \lor (b \land c), (a \lor b) \land (a \lor c)) = A((a \lor b) \land (a \lor c), a \lor (b \land c)) = 1$;
4. $A(a \land (a \lor b), a) = A(a, a \land (a \lor b)) = 1$;
5. $A((a \lor b) \land a, a) = A(a, (a \lor b) \land a) = 1$;
6. $A((a \lor b) \lor b, b) = A(b, (a \lor b) \lor b) = 1$ for all $a, b, c \in R$.

**Example 2.2.** Let $R = \{a, b, c\}$ Define two binary operations $\lor$ and $\land$ on $R$ as follows.

![Diagram](image)

Define a fuzzy relation $A: R \times R \to [0, 1]$ as follows:

$A(a, a) = A(b, b) = A(c, c) = 1,A(b, a) = A(b, c) = A(c, b) = 0,A(a, b) = 0.2$ and $A(a, c) = 0.4$

Clearly $(R, A)$ is a GADFL.

**Definition 2.3.** Let $(R, A)$ be a GADFL. A non-empty subset $I$ of $R$ is said to be an ideal of $(R, A)$, if it satisfies the following conditions:

1) If $x \in R$, $y \in I$ and $A(x, y) > 0$, then $x \in I$;
2) If $x, y \in I$ then $x \lor y \in I$.

**Definition 2.4.** If $I \neq R$, an Ideal $I$ of $(R, A)$ is termed proper. If $F \neq R$, a filter $F$ of $(R, A)$ is considered proper. For each any $x, y \in R$, $x \land y \in P(x \lor y \in P) \Rightarrow x \in P or y \in P$, a suitable ideal (filter) $P$ of $R$ is said to be prime. If $R - P$ is PF, it is obvious that a subset $P$ of $R$ is a PI.

3. Minimal Prime Ideals (MPIs) in GADFL
In this section we study many interesting and important properties of MPIs and MPFs of $L(R_f, A_f)$.
**Definition 3.1**

Let $I_f$ be an ideal of $L(R_f, A_f)$. A PIP$_f$ is said to be a MPI of GADFL belonging to an ideal $I_f$ if
1. $I_f \subseteq P_f$ and
2. There is no PIQ$_f$ such that $I_f \subseteq Q_f \subseteq P_f$. That is $P_f$ is maximal among the PIs of $L(R_f, A_f)$ containing $I_f$.

**Theorem 3.2**

Let $I$ be an ideal of $L(R_f, A_f)$. Let $P_f$ be a PI containing $I_f$. Then $P_f$ is MPI belonging to $I_f$ if and only if for each $a_f \in P_f$ there is $b_f \notin P_f$ such that $a_f \land b_f \in I_f$.

Proof:

Let $P_f$ be a MPI belonging to $I_f$.

Then prove that $R_f - P_f$ is a PF which is negligible in terms of the attribute of not meeting $I_f$.

Let $a_f \in P_f$. Then $a_f \notin R_f - P_f$.

Let $E_f = (R_f - P_f) \lor [a_f]$.

Suppose $E_f \cap I_f = \emptyset$.

Then the PI $P_f$ of $R_f$ is a MPI iff for each $a_f \in P_f$, there is $b_f \notin P_f$ such that $a_f \land b_f = 0$, there is a PF $H_f$ such that $H_f \subseteq E_f$ and $H_f \cap I_f = \emptyset$.

Therefore, $R_f - H_f$ is a PI and $I_f \subseteq R_f - H_f$.

Since $H_f \subseteq E_f \subseteq R_f - P_f$, we get $P_f \subseteq R_f - H_f$ and hence $R_f - H_f = P_f$.

That is $H_f = R_f - P_f$ so that $a \in R_f - P_f$.

This is a contradiction.

Therefore $E_f \cap I_f = \emptyset$.

Choose $r_f \in [(R_f - P_f) \lor [a_f]] \cap I_f$.

Then $t_f \in I_f$ and $t_f \in (R_f - P_f) \lor [a_f]$.

Therefore $t_f = b_f \land y_f$ where $b_f \in R_f - P_f$ and $y_f \in [a_f]$.

Now, $t_f = A_f(b_f \land (y_f \lor a_f), 0)$

$= A_f((b_f \land y_f) \lor (b_f \land a_f), 0)$

$= A_f(t_f \lor (b_f \land a_f), 0)$

$= A_f(t_f \lor 0, 0)$

$= A_f(0, 0) \geq 0$. (Since $b_f \land a_f = 0$ and $a_f \land b_f = 0$).

Therefore $b_f \land a_f \in I_f$.

That is for every $a_f \in P_f$, there is $b_f \notin P_f$ such that $b_f \land a_f \in I_f$.

Let $K_f$ be any PI belonging to $I_f$ and $P_f \subseteq K_f$.

Let $a_f \in P_f$. Then from our assumption there is $b_f \notin P_f$ such that $a_f \land b_f \in P_f$.

Now $P_f \subseteq K_f \Rightarrow a_f \land b_f \in K_f$ and hence $a_f \in K_f$.

Since $b_f \notin K_f$. Therefore $P_f$ is a PI belonging to $I_f$.

Hence the proof.

**Definition 3.3**

A set $S_f$ of $L(R_f, A_f)$ GADFL is said to be multiplicatively closed subset of $L(R_f, A_f)$ if
$S_f \neq \emptyset$ and for any $a_f, b_f \in S_f$ implies $A(a_f \land b_f, 0) > 0$ and $a_f \land b_f \in S_f$.

**Theorem 3.4**

Let $I_f$ be an ideal and $S_f$ be a multiplicatively closed subset of GADFL $L(R_f, A_f)$ such that $I_f \cap S_f = \emptyset$. Then there is a MPI$_I$ of $L(R_f, A_f)$ such that $L - S_f \subseteq T_f \subseteq I_f$.

**Proof:**

Let $I_f$ be an ideal and $S_f$ be a multiplicatively closed subset of GADFL $L(R_f, A_f)$ such that $I_f \cap S_f = \emptyset$.

Then there exists a PIP$_f$ of $L(R_f, A_f)$ such that $P_f \subseteq I_f$ [By the definition of 2.5] and $S_f \cap P_f = \emptyset$.

Since $P_f$ is a PI of $L(R_f, A_f)$, $L - P_f$ is a PF of $L(R_f, A_f)$.

Now, we prove that $L - P_f \subseteq S_f$.

Let $x_f, y_f \in S_f$ then $x_f \land y_f \in S_f$, [By the definition of 2.4]

Now $x_f, y_f \in S_f \Rightarrow A_f\left((x_f \lor (x_f \land y_f), (y_f \lor (x_f \land y_f)), 0) > 0\right)$

$\Rightarrow A_f\left((x_f \lor (x_f \land y_f)), (y_f \lor (x_f \land y_f)), 0\right) > 0$

$\Rightarrow A_f(x_f \land y_f, 0) > 0$

$\Rightarrow A_f(0, 0) > 0$

∴ $x_f, y_f \in S_f \Rightarrow x_f \land y_f \in S_f$

Similarly, we have to prove that $x_f \land y_f \in L - P_f$

∴ we get $L - P_f \subseteq S_f$.

Also, $I_f \cap (L - P_f) = \emptyset$.

Now let $G = \{F_f | F_f$ is a filter of $L(R_f, A_f), F_f \subseteq S_f \text{ and } I_f \cap F_f = \emptyset\}$

Also, let $G = \{x_f \in R | A_f \left((x_f \lor (x_f \land y_f), 0) > 0 \forall y_f \in F_f\right)\}$

clearly, $L - P_f \notin G$ and hence $G \neq \emptyset$.

Therefore, there exists a maximal element $g_f$ and $G_f \in G \forall g_f \in G_f$.

To prove that $G_f$ is a PF of $L(R_f, A_f)$.

Assume $G_f \in G, g_f \in G_f$ and $r_f \in L(R_f, A_f)$.

Since $g_f$ is maximal element, $A_f\left((g_f \lor x_f, 0) > 0 \forall x_f \in R\right)$.

Hence $A_f\left((x_f, (r_f \lor g_f) \land x_f, 0) = A_f\left((x_f, (g_f \land r_f) \land x_f, 0\right)\right.$

$\geq \sup_{x \in R} \min \left\{A_f\left((x_f, k_f\right), A_f\left((x_f, (g_f \land x_f) \lor (r_f \land x_f)\right), 0\right) > 0\right.$

$\geq \min \left\{A_f\left((x_f, g_f \land x_f, 0\right), A_f\left((g_f \land x_f, (g_f \land x_f) \lor (r_f \land x_f))\right), 0\right) > 0\right.$

Hence $A_f\left((x_f, (r_f \lor g_f) \land x_f, 0) > 0\right.$ and it follows that $r_f \lor g_f \in G$ and $G_f \in G$ where $g_f \in G$ and $r_f \in L(R_f, A_f)$.

Thus $G_f$ is a PF of $L(R_f, A_f)$, and hence $L - G_f$ is a PI of $L(R_f, A_f)$.

Also, $L - G_f \subseteq I_f$ and $S_f \cap (L - G_f) = \emptyset$.

Clearly, $G_f \subseteq S_f$, now let, $T_f$ be any other Plof $L(R_f, A_f)$.

To prove that $T_f$ is a MPI of $L(R_f, A_f)$ such that $L - S_f \subseteq T_f \subseteq I_f$.

Let $T_f$ be Plof $L(R_f, A_f)$ such that $T_f \subseteq I_f$ and $L - G_f \subseteq I_f$. 

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This gives \( I_f \cap (L - T_f) = \emptyset \) and also \( L - T_f \subseteq G_f \subseteq S_f \),
but \( g_f \in G_f \) is maximal element of \( G \).
Therefore, we get \( (L - T_f) = G_f \) this gives \( T_f = L - T_f \).
Thus, \( L - S_f \subseteq T_f \subseteq I_f \) (since \( L - T_f \subseteq G_f \subseteq S_f \)).
Therefore, we get \( T_f \) is a MPI of \( L(R_f, A_f) \), such that \( L - S_f \subseteq T_f \subseteq I_f \).
Hence the proof.

**Theorem 3.5**

Let \( I_f, J_f \) be any two ideals of GADFL \( L(R_f, A_f) \). Then \( J_f \) is a MPI belonging to \( I_f \).

Proof:

Let \( I_f, J_f \) be any two ideals of GADFL \( L(R_f, A_f) \).

To prove that \( J \) is MPI belonging to \( I_f \).

Now \( I_f \cap J_f = \emptyset \) implies \( J_f \subseteq I_f \).

Let \( e_f, h_f \in L(R_f, A_f) \) such that \( e_f \notin J_f \) and \( h_f \notin J_f \) then \( e_f, h_f \in L - J_f \).

We have to prove that \( e_f \land h_f \in L - J_f \).

Now, let \( e_f, h_f \in L - J_f \Rightarrow A_f(e_f \lor (e_f \land h_f), h_f \lor (e_f \land h_f), 0) > 0 \)
\[ \Rightarrow A_f((e_f \lor h_f), (e_f \lor h_f) \land h_f, 0) > 0 \]
\[ \Rightarrow A_f((e_f \land h_f), 0) > 0 \] [Since \( e_f \lor h_f = 1 \) and \( e_f \land h_f = 0 \) by the GADFL condition]
\[ \Rightarrow A_f(0, 0) > 0 \]

\[ \therefore e_f, h_f \in L - J_f \Rightarrow e_f \land h_f \in L - J_f \]

Hence \( e_f \land h_f \in L - J_f \).

This gives \( e_f \land h_f \notin L - J_f \)

Therefore \( J_f \) is a PI of GADFL \( L(R_f, A_f) \) and \( J_f \subseteq I_f \).

Let \( K_f \) be any other PI of GADFL \( L(R_f, A_f) \) such that \( K_f \subseteq I_f \) and \( J_f \subseteq K_f \).

Then \( L - K_f \) is a MPI of \( L(R_f, A_f) \).
and \( I_f \cap (L - K_f) = \emptyset \).

Therefore, we get \( L - J_f \subseteq L - K_f \) and hence \( K_f \subseteq J_f \).
This gives \( K_f = J_f \).

Therefore \( J_f \) is a MPI of GADFL \( L(R_f, A_f) \).
Hence \( J_f \) is a MPI belonging to \( I_f \).

Hence the proof.

Now we prove the characterization theorems for Minimal Prime Ideals (MPIs).

**Theorem 3.6**

Let \( L(R_f, A_f) \) be a GADFL. Then the following are equivalent.
1. \( L(R_f, A_f) \) be an ADFL.

2. Every MPI is an ideal.

3. Every \( PIP_f \) with \( P_f \cap D_f = \emptyset \) is an ideal.

Proof:

Assume (1) ⇒ (2). Let \( L(R_f, A_f) \) be an ADFL.

To prove that every MPI is an ideal.

Let \( P_f \) is any non-empty subset and \( P_f^* \) is a MPI, such that \( P_f^* \subseteq P_f \).

The set \( P_f^* = \{ x_f \in R_f | A_f(x_f \land p_f, 0) \geq 0 \text{ for all } p_f \in P_f \} \), now let \( a_f, b_f \in P_f^* \).

Then \( A_A_f(a_f \land p_f, 0) > 0 \) and \( A_f(b_f \land p_f, 0) > 0 \) \( \forall p_f \in P_f \), on the other hand, since

\[ L(R_f, A_f) \text{ is a GADFL}, \]

\[ A_f(0, 0, 0) > 0 \] \( \forall p_f \in P_f \)

Then

\[ A_f((a_f \lor b_f) \land p_f, 0) = A_f((a_f \land p_f) \lor (b_f \land p_f), 0) \]

\[ = A_f(0 \lor 0, 0) \]

\[ = A_f(0, 0, 0) > 0 \]

Thus, \( a_f \lor b_f \in P_f^* \)

Again let \( a_f \in P_f^* \) and \( x_f \in R_f \)

Then \( A_f(a_f \land p_f, 0) > 0 \) \( \forall p_f \in P_f \)

Now, \( A_f((a_f \land x_f) \land p_f, 0) = A_f((x_f \land a_f) \land p_f, 0) \)

\[ = A_f((x_f \land a_f) \land p_f, 0) \]

\[ \geq \sup_{y \in R} \min \{ A_f(x_f \land (a_f \land p_f), y_f), A_f(y_f, 0) \} \]

\[ \geq \min \{ A_f(x_f \land (a_f \land p_f), a_f \land p_f), A_f(a_f \land p_f, 0) > 0 \} \forall p_f \in P_f \]

Hence \( (a_f \land x_f) \land p_f \in P_f^* \forall p_f \in P_f \).

Thus \( P_f^* \) is an ideal of \( L(R_f, A_f) \) GADFL.

Assume (2) ⇒ (3)

Assume that every MPI is an ideal.

To prove that every PI \( P_f \) with \( P_f \cap D_f = \emptyset \) is an ideal.

Let \( D_f \) be a PI of \( L(R_f, A_f), L-D \) is a filter of \( L(R, A) \) and \( L-D \subseteq P_f \).

Also \( P_f \cap D_f = \emptyset \) is an ideal.

Assume (3) ⇒ (1)

Let \( n_f \in R_f, P_f \) is a PI with \( P_f \cap D_f = \emptyset \) is an ideal.

Let \( n_f, z_f \in R \) since \( A_f(n_f \land z_f, (n_f \land z_f) \land z_f) = A_f((n_f \land z_f) \land z_f, n_f \land z_f) = 1 \)

then \( (n_f, n_f \land z_f) \in P_f \). Also \( A_f(z_f \land z_f, z_f \land z_f) = 1 \)

Hence \( A_f(z_f, z_f) \in P_f \).

Since \( P_f \) is a PI with \( P_f \cap D_f = \emptyset \) is an ideal of \( L(R_f, A_f) \).

Hence\( A_f((n_f \lor z_f) \land z_f, [(n_f \land z_f) \lor z_f] \land z_f) \)

\[ = A_f(((n_f \lor z_f) \lor z_f) \land z_f, (n_f \lor z_f) \land z_f) = 1 \]

\[ \Rightarrow A_f((n_f \lor z_f) \land z_f, z_f \land z_f) = A_f(z_f \lor z_f, (n_f \lor z_f) \land z_f) = 1 \]
\[ A_f \left( (n_f \lor z_f) \land z_f, z_f \right) > 0 \quad \text{and} \quad A_f \left( z_f, (n_f \lor z_f) \land z_f \right) > 0 \]

\[ L(R_f, A_f) \text{is an Almost Distributive Fuzzy Lattice.} \]

Hence \( L(R_f, A_f) \) is an ADFL. Hence the proof.

**Definition 3.7**

Let \( F_f \) be a filter of \( L(R_f, A_f) \). A PFG\(_f\) is named to be a MPF of GADFL be in the right place to a filter \( F_f \) if

1. \( F_f \supseteq G_f \) and

2. there is no PFH such that \( F_f \supseteq H \supseteq G_f \). That is \( G_f \) is minimal among the PFs of \( L(R, A) \) covering \( F_f \).

**Theorem 3.8**

Every PI of a GADFL \( L(R_f, A_f) \) contains an MPI.

Proof:

Let \( P \) be a PI of \( L(R_f, A_f) \). Let \( F_f = L - P_f \). Then \( F_f \) is a PF of \( L(R_f, A_f) \) such that \( F_f \cap L - P_f \neq \emptyset \) there is a maximal filter \( G \) in \( L(R_f, A_f) \).

Now \( F_f \supseteq G_f \)

To prove \( L - G_f \supseteq L - F_f \)

Let \( x_f, y_f \in L - G \) then there exist \( a_f, b_f \in L(R_f, A_f) \) and \( c_f, d_f \in G \) such that \( a_f \land c_f = 0 \) and \( b_f \land d_f = 0 \).

Then \( x_f \land y_f \in L - G_f \)

Now \( x_f, y_f \in L - G_f \Rightarrow A_f(x_f \land y_f, (a_f \land c_f) \lor (b_f \land d_f), 0) \)

\[ = A_f \left( x_f, y_f, \left( (a_f \land c_f) \lor (x_f \land y_f) \right) \lor \left( (b_f \land d_f) \land (x_f \land y_f) \right), 0 \right) \]

\[ = A_f \left( x_f, y_f, (0 \land (x_f \land y_f)) \lor (0 \land (x_f \land y_f)), 0 \right) \]

\[ = A_f \left( x_f, y_f, 0, 0 \right) \]

\[ = A_f \left( x_f, y_f, 0 \right) > 0 \]

\[ \therefore x_f \land y_f \in L - G_f \text{ implies } x_f \land y_f \in L - G_f \]

Similarly, \( x_f \land y_f \in L - F_f \)

Hence \( L - G_f \supseteq L - F_f = P_f \)

Therefore \( P_f \) is a MPI of \( L(R_f, A_f) \) if and only if is aMPF. Therefore, we get \( L - G_f \) is a MPI contained in \( P_f \).

**Theorem 3.9**

Every ideal of GADFL \( L(R_f, A_f) \) is the union of all MPIs containing it.

Proof:

Let \( I_f \) be an ideal of GADFL \( L(R_f, A_f) \)

Then \( I_f = \{ x_f \in R_f | A_f(x_f, s_f, 0) > 0 \land s_f \in I_f \} \)
Now let \( I_0 = \bigcup \{ P_f \mid P_f \text{ is a MPI containing } I_f \} \). Clearly \( I_f \supseteq I_0 \) --------(1)

Conversely, Let \( a_f \notin I_f \)

Then \( x \land s_f \neq 0 \) for all \( s_f \in I_f \)

Hence \( x_f \notin P_f \) and \( s_f \notin P_f \).

Since \( P_f \) is prime \( [s_f]^* \supseteq P_f \forall s_f \in I_f \)

Therefore \( I_f \supseteq P_f \)

Thus \( P_f \) is a MPI containing \( I_f \) and \( x_f \notin P_f \). Therefore, we get \( x_f \notin I_0 \) which yields that

\[ I_f \supseteq I_f \] ------(2)

From equations (1) and (2) we get \( I_f = I_0 \).

4. Conclusion

The ideas of minimum prime ideals in GADFL are described in this work and numerous feature of minimal prime ideals are examined. Examine the characteristics of minimal prime ideals provided in this study in further depth. Finding the \( S \) – Ideals in Dual of GADFLs is an exciting future project. On \( S \) – Ideals in Dual of GADFL, we will also derive certain characterization theorems.

References